Potential Hasse principle violations for Châtelet surfaces

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Motivation

Let X be a smooth projective geometrically integral variety over $\mathbb{Q}$.

Question

Is $X(\mathbb{Q}) = \emptyset$?

Definition (Adélic points)

$$X(\mathbb{A}_{\mathbb{Q}}) := X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$$

Since $X(\mathbb{Q}) \subset X(\mathbb{A}_{\mathbb{Q}})$, if $X(\mathbb{A}_{\mathbb{Q}}) = \emptyset$ then $X(\mathbb{Q}) = \emptyset$.

Question

Do there exist varieties where $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$?

YES! Such varieties are said to fail the **Hasse principle** (HP).

When can X fail the Hasse principle?

Consider the equation $y^2 - az^2 = P(\lambda)$ where $a \notin \mathbb{Q}^{\times 2}$ and $P \in \mathbb{Q}[\lambda]$ is separable of degree 4.

1) $y^2 - az^2 = (\lambda - 3)(\lambda^3 + 17\lambda - 3)$; $(y, z, \lambda) = (0, 0, 3)$

2) $y^2 - az^2 = (\lambda - 57)(\lambda + 163)(\lambda^2 - 13)$; $(y, z, \lambda) = (0, 0, -163)$

3) $y^2 - az^2 = (3 - \lambda^2)(\lambda^2 - 2)$; $(y, z, \lambda) =$??

4) $y^2 - az^2 = 5\lambda^4 - 17\lambda^2 + 2\lambda - 15$; $(y, z, \lambda) =$???

Upshot: The only factorization of P that could produce an equation which may fail HP is 3).

How could one show that X fails the Hasse principle?

$X(\mathbb{A}_{\mathbb{Q}})$ is computable, $X(\mathbb{Q})$ is not in general.

Idea

Find some computable set T such that

$$X(\mathbb{Q}) \subseteq T \subseteq X(\mathbb{A}_{\mathbb{Q}})$$

and show that $T = \emptyset$.

Using the Brauer group of X, Manin (1971) defined the Brauer-Manin set:

$$X(\mathbb{Q}) \subseteq X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \subseteq X(\mathbb{A}_{\mathbb{Q}}).$$

Thus, the Brauer-Manin set can obstruct the existence of rational points.

Outline

1) Brauer groups
2) Brauer-Manin set
3) A useful isomorphism
4) Châtelet Surfaces
5) Main theorem
6) An example!
7) Open questions

The Brauer group of a field

Definition

Let k be a field. A central simple k-algebra (CSA/k) $\mathcal{A}$ is

- *a finite-dimensional k-algebra*
- *center is k*
- *has no non-trivial proper two-sided ideals*

Brauer equivalence

Let $\mathcal{A}, \mathcal{A}'$ be two central simple k-algebras. $\mathcal{A}$ and $\mathcal{A}'$ are **Brauer equivalent** if

$$\mathcal{A} \otimes_k \mathrm{M}_n(k) \cong \mathcal{A}' \otimes_k \mathrm{M}_m(k)$$

for some $n, m \in \mathbb{Z}_{>0}$.

The Brauer group of a field

We then define the Brauer group of k

$$\operatorname{Br} k := \frac{\{CSA/k\}}{\text{Brauer equivalence}}$$

Examples

1) If k is algebraically closed, then $\operatorname{Br} k = 0$.

2) Let $a, b \in k^\times$. A (generalized) quaternion algebra, (a, b), is the k-vector space with basis $\{1, i, j, ij\}$ where $i^2 = a, j^2 = b$ and $ji = -ij$.

3) $\operatorname{Br} \mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$ with unique non-trivial element represented by the Hamiltonian quaternions $(-1, -1)$.

Brauer group of a scheme

Let X be a *nice* (smooth projective geometrically integral) variety over number field k.

Definition

We define the Brauer group of X to be

$$\operatorname{Br} X := \mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$$

This agrees with the definition of $\operatorname{Br} k$ in that $\operatorname{Br}(\operatorname{Spec} k) = \operatorname{Br} k$.

We define $X(\mathbb{A}_k)^{\mathrm{Br}}$ using the Brauer group of X.

Brauer-Manin set

Let k be a number field, Ω_k the set of places of k, and k_v the completion of k at $v \in \Omega_k$.

- Given $P \in X(k)$ the map $P\colon \operatorname{Spec} k \to X$ induces a map on the Brauer group $P^*\colon \operatorname{Br} X \to \operatorname{Br} k$.
- Fix $\mathcal{A} \in \operatorname{Br} X$ and define $\operatorname{ev}_{\mathcal{A}} : X(k) \to \operatorname{Br} k$ to be the image of $\mathcal{A}$ under P^*.

For $\mathcal{A} \in \operatorname{Br} X$, we obtain a commuting diagram

$$\begin{array}{c} \prod_{v \in \Omega_v} X(k_v) := X(\mathbb{A}_k) \\ \Big\downarrow \operatorname{ev}_{\mathcal{A}} \\ \prod_{v \in \Omega_k} \operatorname{Br} k_v \end{array}$$

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This picture shows that $X(k) \subset X(\mathbb{A}_k)^{\mathcal{A}}$ where

$$X(\mathbb{A}_k)^{\mathcal{A}} = \left\{ (P_v) \in X(\mathbb{A}_k) \colon \ \sum_{v \in \Omega_k} \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0 \right\}$$

We call

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \bigcap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbb{A}_k)^{\mathcal{A}}$$

the *Brauer-Manin set*.

Brauer-Manin obstruction

Proposition (Manin 1971)

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)$$

If $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ we say there is a **Brauer-Manin obstruction to the Hasse principle**.

Question

How does one effectively compute $X(\mathbb{A}_k)^{\mathrm{Br}}$? In other words, how do I find the "most relevant" $\mathcal{A} \in \operatorname{Br} X$ to show that $X(\mathbb{A}_k)^{\mathcal{A}} = \emptyset$, if it exists?

Proposition

To compute $X(\mathbb{A}_k)^{\mathrm{Br}}$ it is enough to compute the intersection over a set of representatives of $\frac{\operatorname{Br} X}{\operatorname{Br} k}$.

A useful isomorphism

- X/k a nice geometrically rational variety.
- $\overline{k}$ be a fixed algebraic closure of k.

$$\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$$

- $\operatorname{Br}_0 X := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X)$.

The Hochschild-Serre spectral sequence in étale cohomology gives the isomorphism

$$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X} \cong \mathrm{H}^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} \overline{X})$$

Lemma

1) If $X(\mathbb{A}_k) \neq \emptyset$ then the map $\operatorname{Br} k \to \operatorname{Br} X$ is injective, hence $\operatorname{Br}_0 X = \operatorname{Br} k$

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Châtelet surfaces

Let $P(\lambda) \in k[\lambda]$ be a separable polynomial of degree 4 and let $a \in k^\times$.

$$X_1 := \operatorname{Proj} \frac{k[\lambda][y,z,t]}{(y^2 - az^2 - P(\lambda)t^2)} \hookrightarrow \mathbb{P}^2_{\mathbb{A}^1_k} \hookleftarrow X_2 := \operatorname{Proj} \frac{k[\mu][Y,Z,T]}{(Y^2 - aZ^2 - Q(\mu)T^2)}$$

with coordinates $(y : z : t, \lambda)$ and $(Y : Z : T, \mu)$ respectively and $Q(\mu) = \mu^4 P\left(\frac{1}{\mu}\right)$.

Let X be the surface obtained by gluing X_1 and X_2 via the isomorphism

$$X_1 - \{\lambda = 0\} \xrightarrow{\sim} X_2 - \{\mu = 0\}$$

$$(y : z : t, \lambda) \mapsto (Y : Z : \mu^2 T, 1/\mu)$$

Châtelet surfaces

Definition

We call X the **Châtelet surface** *given by* $y^2 - az^2 = P(\lambda)$.

It is smooth, projective, geometrically integral and...

Proposition

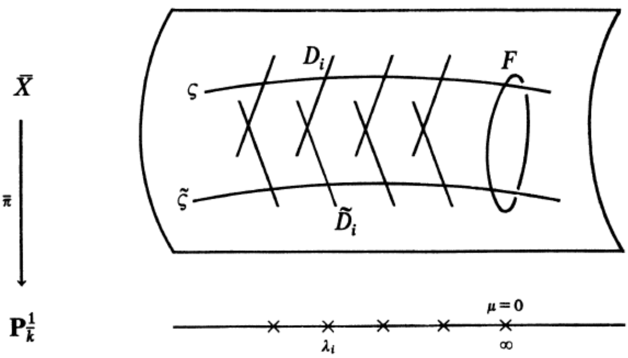
X is geometrically rational

Upon base changing to $\overline{k}$ we obtain a very nice picture. Let

$$P(\lambda) = c\prod_{i=1}^{4}(\lambda - \lambda_i)$$

Châtelet surfaces

$$y^2 - az^2 = P(\lambda)$$



[Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1987]

Recap

Goal

Find a Châtelet surface $X/\mathbb{Q}$ that fails the Hasse principle.

- $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \left\{(P_v) \in X(\mathbb{A}_{\mathbb{Q}}) \colon \sum_{v \in \Omega_{\mathbb{Q}}} \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0\right\}$.
- $X(\mathbb{A}_k)^{\mathrm{Br}} = \bigcap_{\text{generators } \mathcal{A} \text{ of } \frac{\mathrm{Br}\, X}{\mathrm{Br}\, k}} X(\mathbb{A}_k)^{\mathcal{A}}$
- $\frac{\mathrm{Br}\, X}{\mathrm{Br}\, k} \cong \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\, \overline{X})$ (and this depends on the factorization of $P(\lambda)$).

Question

If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ could X still violate HP?

Main Theorems

Theorem (Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1987)

Let k be a number field and X/k a Châtelet surface. X fails the Hasse principle if and only if there is a Brauer-Manin obstruction.

Theorem

Let X be the Châtelet surface given by $y^2 - az^2 = P(\lambda)$. $\frac{\mathrm{Br}\,X}{\mathrm{Br}\,k}$ depends on the factorization of P and is given by

$$H^1(\mathrm{Gal}(\overline{k}/k), Pic(\overline{X})) = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & P \text{ has four rational roots} \\ \mathbb{Z}/2\mathbb{Z} & P \text{ has an irred. deg. 2 factor} \\ \{0\} & \text{otherwise} \end{cases}$$

An Example

Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 + z^2 = (\lambda^2 - 2)(3 - \lambda^2)$$

Theorem (Iskovskikh 1971)

X fails the Hasse principle

Sketch

Show:

1) $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$

2) $\alpha = (-1, 3 - \lambda^2) \in \operatorname{Br} \mathbf{k}(X)$

3) $X(\mathbb{A}_{\mathbb{Q}})^{\alpha} = \emptyset$

How do I compute $X(\mathbb{A}_{\mathbb{Q}})^\alpha$?

$$X(\mathbb{A}_{\mathbb{Q}})^\alpha = \left\{ P \in X(\mathbb{A}_{\mathbb{Q}}) : \sum_{v \in \Omega_{\mathbb{Q}}} \mathrm{inv}_v(\mathrm{ev}_\alpha(P)) = 0 \right\}$$

Compute $\mathrm{inv}_v(\mathrm{ev}_\alpha(P))$ for each $v \in \Omega_{\mathbb{Q}}$.

Lemma

Let k be a local field and X/k a smooth variety. Let $U \subset X$ be a nonempty Zariski open set of X. Then $U(k)$ is analytically dense in $X(k)$.

Lemma

$\mathrm{inv}_v \circ \mathrm{ev}_\alpha$ *is a continuous function on* $X(\mathbb{Q}_p)$

Corollary

Let X_0 be the affine surface given by $y^2 + z^2 = (\lambda^2 - 2)(3 - \lambda^2)$. To compute $X(\mathbb{A}_{\mathbb{Q}})^\alpha$, it is enough to evaluate at points $P \in X_0(\mathbb{Q}_p)$.

An Example

Proposition

Fix a place v_p of $\mathbb{Q}$. Then for any $P \in X_0(\mathbb{Q}_p)$,

$$\mathrm{inv}_{v_p}(\mathrm{ev}_\alpha(P)) = \begin{cases} 0 & p \neq 2 \\ \frac{1}{2} & p = 2 \end{cases}$$

Corollary

$$X(\mathbb{A}_\mathbb{Q})^\alpha = \emptyset$$

X fails the Hasse principle.

Open questions

Question

What can one say about nice varieties $X/\mathbb{Q}$ with $X(\mathbb{A}_{\mathbb{Q}}) = \emptyset$? Do they fail HP over some extension of $\mathbb{Q}$?

Definition

Given a nice variety $X/\mathbb{Q}$ such that $X(\mathbb{A}_{\mathbb{Q}}) = \emptyset$ we say X is a **potential Hasse principle** *(PHP)* **violation** *if there exists an extension $L/\mathbb{Q}$ such that $X(L) = \emptyset$ and $X(\mathbb{A}_L) \neq \emptyset$*

Open questions

Conjecture (Clark 2011)

Let k be a global field. Every curve C/k of genus ≥ 2 with $X(k) = \emptyset$ is a PHP violation

Question

Is every Châtelet surface $X/\mathbb{Q}$ with $X(\mathbb{A}_{\mathbb{Q}}) = \emptyset$ a PHP violation.

Open questions

Theorem (Creutz-Viray, 2020)

Let X/k be a Châtelet surface.

- *$\exists$ finite set of places $S \subseteq \Omega_k$*
- *A set of local quadratic extensions L_v/k_v for all $v \in S$ such that*
- *F/k is a quadratic extension with $F_v = L_v$ for all $v \in S$*

$$X(\mathbb{A}_F)^{\mathrm{Br}} \neq \emptyset$$

Future Goals

Can one characterize the Châtelet surfaces that are PHP violations?

THANK YOU!