POTENTIAL HASSE PRINCIPLE VIOLATIONS FOR CHÂTELET SURFACES

SAM ROVEN

Abstract. This paper shows how to construct a Châtelet surface which has a Brauer-Manin obstruction to the Hasse principle. Also, we discuss open questions regarding Châtelet surfaces that fail the Hasse principle.

Contents

1. Introduction

For $\mathcal{A} \in \operatorname{Br} X$, we obtain a commuting diagram

$$\begin{array}{ccccccccc} & & X(k) & \longrightarrow & X(\mathbb{A}_k) & & & & \\ & & \big\downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \big\downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & & & \\ 0 & \longrightarrow & \operatorname{Br} k & \xrightarrow{\phi} & \bigoplus_{v\in\Omega_k} \operatorname{Br} k_v & \xrightarrow{\sum_v \mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Let X be a smooth geometrically integral variety over a number field k. We say X satisfies the Hasse principle if the set $X(k)$ of k-rational points is non-empty whenever the set of adélic points $X(\mathbb{A}_k)$ is also non-empty. If X fails the Hasse principle, it is natural to ask about the obstructions that account for this failure. In 1970, Manin used the Brauer group of X to define the **Brauer-Manin set** $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. It was proven [Man71] that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$ hence this set can obstruct the existence of k-points on X. This obstruction is known as the **Brauer-Manin obstruction**.

A wide area of research investigates the extent with which sets like $X(\mathbb{A}_k)^{\mathrm{Br}}$ give obstructions to the Hasse principle. In particular one can fix numerical invariants, like the dimension of X, and attempt to classify when the Brauer-Manin set explains this failure. In 1971, Iskovskikh [Isk71] constructed an example of a smooth projective surface that failed the Hasse principle. Years later, in the landmark paper [CTSSD87], Colliot-Thélène, Sansuc, and Swinnerton-Dyer showed that the Brauer-Manin obstruction explains all failures of the Hasse principle for a class of surfaces known as Châtelet surfaces, which are surfaces that contain an affine open subscheme cut out by $y^2 - az^2 = P(\lambda)$ with P a separable degree 4 polynomial. In this paper, we explain how one can use the Brauer group of a Châtelet surface (modulo constant algebras) to give a Brauer-Manin obstruction to the Hasse principle. In particular we prove the following theorem

Theorem 1. *Let L denote the splitting field of P so that $L(\sqrt{a})$ is the splitting field of X. Assume $a \notin L^{\times 2}$. The Brauer group of X depends on the factorization of $P(\lambda)$ with Brauer groups given by*

$$H^1(G_k, Pic(\overline{X})) = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & P(\lambda) \textit{has four rational roots} \\ \mathbb{Z}/2\mathbb{Z} & P(\lambda) \textit{has one irredicible quadratic factor} \\ \{0\} & \textit{otherwise} \end{cases}$$

Iskovskikh's original example was indeed a Châtelet surface, but his proof that this Châtelet surface had no $\mathbb{Q}$-points only used methods based on quadratic reciprocity. After proving the above theorem, we give the example of Iskovskikh but use the Brauer-Manin obstruction to explain the failure of the Hasse principle over $\mathbb{Q}$. We finish the paper by giving a brief survey of recent results concerning Châtelet surfaces, along with currently open questions that the author intends to answer.

1.1. **Outline.** In section 2, we introduce quaternion algebras and give some results that allow one to determine when they are split. We then generalize to all finite dimensional central simple k-algebras and define the Brauer group of a field. We finish off section 2 with characterizations of Brauer groups of local and global fields. In section 3 we define the Brauer group of a scheme, as well as the Brauer-Manin set, proving several related and crucial results along the way. We then use the Hochschild-Serre spectral sequence to give an isomorphism that will enable us to find Brauer classes which give a Brauer-Manin obstruction to a Châtelet surface X. In section 4 we define Châtelet surfaces and give an overview of results concerning the classification of ruled surfaces that we then use to prove that Châtelet surfaces are geometrically rational. We finish off this section by giving explicit descriptions of the Picard group and intersection theory on a Châtelet surface. We state and prove the main results of this paper in section 5, proving the theorem stated above, and as mentioned, we outline open questions regarding Châtelet surfaces that fail the Hasse principle over extensions of the ground field.

1.2. **Notation.** Throughout this paper, k will always denote a field of characteristic 0 and all the k-algebras we consider are finite dimensional. We use $\overline{k}$ to denote a fixed algebraic (hence separable) closure of k. If k is a global field, we let $\mathbb{A}_k$ denote the adéle ring of k and Ω_k the set of places of k. For a fixed $v \in \Omega_k$, we let k_v denote the completion of k at v, let $\mathcal{O}_v$ denote the valuation ring in k_v, and $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$.

For a field k let $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$ denote the absolute Galois group of k. Given a scheme X over k, and an extension L/k, we write $X_L \coloneqq X \times_{\mathrm{Spec\ k}} \mathrm{Spec}\, L$ and $\overline{X} \coloneqq X_{\overline{k}}$. We also let $X(k)$ be the k-points of X and $X(\mathbb{A}_k)$ the adélic points of X.

By a k-variety we mean a separated scheme of finite type over k and by nice k-variety we mean a smooth projective geometrically integral k-variety. By surface we mean a smooth projective variety of dimension 2.

2. Brauer Groups

2.1. Quaternion Algebras.

Definition 1. (Quaternion Algebra) For any two elements $a, b \in k^\times$ the *(generalized) quaternion algebra* (a, b) is the 4-dimensional k-algebra with basis $1, i, j, ij$, and multiplication being determined by

$$i^2 = a,\ j^2 = b,\ ji = -ij$$

Remark 1. The isomorphism class of the algebra (a, b) depends only on the classes of a and b in $k^\times/k^{\times 2}$. The substitution $i \mapsto ui, j \mapsto vj$ induces an isomorphism $(a, b) \cong (u^2a, v^2b)$ for all $u, v \in k^\times$.

In particular, taking the map $i \mapsto abj, j \mapsto abi$ we get

$$(a, b) \cong (a^2b^3, a^3b^2) \cong (b, a)$$

Remark 2. The assignment

$$i \mapsto I \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ j \mapsto J \coloneqq \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$$

defines an isomorphism $(1, b) \cong M_2(k)$, because the matrices

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix},\ IJ = \begin{bmatrix} 0 & b \\ -1 & 0 \end{bmatrix}$$

generate $M_2(k)$ as a k-vector space and satisfy the relations

$$I^2 = \mathrm{Id}, \quad J^2 = b\,\mathrm{Id}, \quad IJ = -JI$$

Given a quaternion algebra (a, b), we say that (a, b) is **split** if $(a, b) \cong \mathrm{M}_2(k)$. Identfying when a quaternion algebra is split will be of central importance to us and it can be done in a a number of ways. Before giving the main result of this section, we give one more definition

Definition 2. (**The Associated Conic**) Given a quaternion algebra (a, b), we define the associated conic, $C(a, b)$, to be the projective plane curve defined by the homogeneous equation

$$ax^2 + by^2 = z^2$$

where x, y, z are homogeneous coordinates in $\mathbb{P}^2$. In the case of $(1, 1) \cong M_2(k)$ we get the circle $x^2 + y^2 = z^2$.

The following result will be our primary tool.

Proposition 1. *For a quaternion algebra (a, b), the following are equivalent.*
(1) (a, b) is split
(2) (a, b) is not a division algebra
(3) The norm map $N : (a, b) \to k$ defined by $N(q) = q\overline{q}$ has a nontrivial zero.
(4) The element b is a norm from the field extension $k(\sqrt{a})/k$.
(5) $C(a, b)$ has a k-rational point

Proof. The implication (1) $\implies$ (2) is obvious. To prove (2) $\implies$ (3), assume that for all $q \neq 0$, $N(q) \neq 0$. Then $\overline{q}/N(q)$ is an inverse to q.
Next, assume (3) and that $a \notin k^{\times 2}$ else the result is obvious. If $q = x + yi + zj + wij$ then $N(q) = x^2 - ay^2 - bz^2 + abw^2 = 0$. This implies that $x^2 - ay^2 = b(z^2 - aw^2) = b(z + \sqrt{a}w)(z - \sqrt{a}w)$, hence

$$b = \frac{N(x + \sqrt{a}y)}{N(z + \sqrt{a}w)}$$

and so (4) follows from the fact that the norm is multiplicative.
Next we show (4) $\implies$ (1) and then separately, that (4) $\Leftrightarrow$ (5). To deduce (1), we show that $(a, b) \cong (1, 4a^2)$. Assuming again that a is not a square in k, if b is a norm from $k(\sqrt{a})$ then so is b^{-1} thus we can find $x, y \in k$ such that $b^{-1} = x^2 - ay^2$. Setting $u = xj + yij$ we have $u^2 = bx^2 - aby^2 = bb^{-1} = 1$. Furthermore, one can check that $ui = -iu$. Setting $v = (1 + a)i + (1 - a)ui$ we can check that $uv = (1 + a)ui + (1 - a)i = -vu$ and $v^2 = (1 + a)^2 a - (1 - a)^2 a = 4a^2$. Note that by considering any non-trivial linear combination $z_1 + z_2 u + z_3 v + z_4 uv = 0$, it is easy to show that $z_1 = z_2 = z_3 = z_4 = 0$ using that fact that $\{1, i, j, ij\}$ is a basis. This implies that $\{1, u, v, uv\}$ is a quaternion basis, and by changing to this basis, we obtain the required isomorphism $(a, b) \cong (1, 4a^2)$ for which (a, b) is split by remark 2 above.
Finally, assuming (4) again we have $x, y \in k$ such that $b = x^2 - ay^2$. From here it is immediate that $(y, 1, x)$ is a k-point on $C(a, b)$. For the converse, if there exist (x_0, y_0, z_0) satisfying $a(x_0)^2 + b(y_0)^2 = z_0^2$ we first observe that we can take $y_0 \neq 0$ else we show that a is a square and are done. This means we can multiply by y_0^{-1} and find that $b = (\frac{z_0}{y_0})^2 - a(\frac{x_0}{y_0})^2$, and (4) is satisfied. If $y_0 = 0$ then we must have $x_0 \neq 0$ otherwise b is a square and we are done. Multiplying by x_0^{-1} we find that a is a norm from $k(\sqrt{b})$. $\square$

Quaternion algebras are a special case of a more general class of algebras known as central simple algebras. We breifly mention some classical results concerning central simple algebras below. For further reading see [GS17, Chapters 1 and 2] and [Voi19, Part I]

Definition 3. A k-algebra is called *simple* if it has no non-trivial (two-sided) ideals. A k-algebra is *central* if its center equals k. A *central simple algebra* is a k-algebra that is both central and simple.

Theorem 2. ***(Wedderburn)***[GS17, Theorem 2.1.3]
Let A be a finite-dimensional simple algebra over k. Then there exists an integer $n \geq 1$ and a division algebra $D \supset k$ such that $A \cong \mathrm{M}_n(D)$. Moreover, the division algebra D is uniquely determined up to isomorphism.

If we take our finite-dimensional simple algebra to also be central, we can say more.

Theorem 3. [GS17, Corollary 2.2.12] *A finite-dimensional k-algebra A is a central simple algebra if and only if there exists an integer $n > 0$ and a finite Galois extension L/k such that $A \otimes_k L$ is isomorphic to the matrix algebra $\mathrm{M}_n(L)$.*

This theorem implies that for any central simple algebra A/k, $\dim_k(A)$ is a square.

Definition 4. **(Splitting field and degree of a CSA)** A field extension L/k for which $A \otimes_k L \cong M_n(L)$ is called a *splitting field* for A. The integer $\sqrt{\dim_k(A)}$ is called the *degree* of A.

Proposition 2. [GS17, Prop 2.2.9)] *Let A be a central simple algebra over k. There is a canonical isomorphism $A \otimes_k A^{\mathrm{opp}} \cong \mathrm{End}_k(A) \cong \mathrm{M}_n(k)$, where n is the degree of A.*

We now have all the tools needed to define the Brauer group of a field.

2.2. **The Brauer Group of a Field.** Since central simple algebras over a field k can be characterized by those algebras A for which there exists a finite Galois extension L/k and an integer $n > 1$ such that $A \otimes_k L \cong M_n(L)$, we can define the following set.

Let $\mathrm{CSA}_L(n)$ denote the set of k- isomorphism classes of central simple k-algebras of degree n split by L. We regard it as a pointed set with the base point being the class of $M_n(k)$.

Two central simple k-algebras A and B are *Brauer equivalent* if $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$ for some $m, n > 0$.

Definition 5. **(The Brauer group of a field)** Brauer equivalence defines an equivalence relation on the union of the sets $\mathrm{CSA}_L(n)$. We denote the set of equivalence classes by $\mathrm{Br}(L/k)$ and define the Brauer group, denoted $\mathrm{Br}\, k$, to be

$$\mathrm{Br}\, k := \bigcup_{L/k \text{ finite Galois}} \mathrm{Br}(L/k)$$

We note here that for any fixed extension L/k, we will refer to $\mathrm{Br}(L/k)$ as the *relative* Brauer group (of the extension L/k). It can alternatively be defined as the kernel of the homomorphism

$$\mathrm{Br}(L/k) = \ker(\mathrm{Br}\ k \to \mathrm{Br}\ L), \quad \mathcal{A} \mapsto \mathcal{A} \otimes_k L$$

Proposition 3. *The set $\mathrm{Br}\, k$ forms an abelian group under tensor product.*

Proof. Basic properties of tensor product imply that the binary operation is commutative and associative. Clearly the identity element is the class of $\mathrm{M}_n(k)$. Moreover, Proposition 1 implies that given a class in $\mathrm{Br}(L/k)$ represented by A, the class of the opposite algebra A^{opp} yields an inverse. □

Equivalently, [Ser91, Chapter X, section 5] one can define the Brauer group via Galois cohomology

$$\mathrm{Br}\ k := \mathrm{H}^2(G_k, \overline{k}^\times)$$

Remark 3. Generalizing the notion of splitting from Remark 2, a Brauer class $\mathcal{A} \in \mathrm{Br}\ k$ is *split by* L if $\mathcal{A}$ is contained in the subgroup $\mathrm{Br}(L/k)$. We also note that Br is a covariant functor from the category of fields to the category of abelian groups.

Remark 4. We can see that each non-trivial Brauer class contains (up to isomorphism) a unique division algebra and Br (L/k) classifies division algebras split by L. To see why, let $\mathcal{A}, \mathcal{A}'$ be central simple k-algebras, so by Theorem 2

$$\mathcal{A} \cong \mathrm{M}_n(D), \ \ \mathcal{A}' \cong \mathrm{M}_{n'}(D')$$

for some division algebras D and D'. If $\mathcal{A}$ and $\mathcal{A}'$ are Brauer equivalent then

$$\mathcal{A} \otimes \mathrm{M}_m(k) \cong \mathcal{A}' \otimes \mathrm{M}_{m'}(k) \implies \mathrm{M}_{nm}(D) \cong \mathrm{M}_{n'm'}(D')$$

From here, uniqueness of the division algebra implies that $D \cong D'$

Remark 5. It follows from Theorem 2 and the previous remark that if $\mathcal{A} \sim_{\mathrm{Br}} \mathcal{A}'$ and $\dim_k(\mathcal{A}) = \dim_k(\mathcal{A}')$, then $\mathcal{A} \cong \mathcal{A}'$, so each Brauer class contains exactly one central simple algebra of fixed degree.

In the case where our ground field is algebraically closed, the Brauer group becomes trivial.

Lemma 1. *Let $k = \overline{k}$. Then every central simple k-algebra is isomorphic to $M_n(k)$ for some $n \geq 1$, hence* $\mathrm{Br}\, k = 0$.

Proof. By the remark, it is enough to show that the only finite-dimensional division algebra $D \supset k$ is k itself. Take any $d \in D$ and consider the extension $k[d]$. Since D is finite dimensional over k, d is algebraic. Since $k = \overline{k}$, $k[d] = k$. □

Lastly, from the cohomological definition of Br k, we can also obtain a definition of the relative Brauer group of a Galois extension L/k.

Proposition 4. $\mathrm{Br}(L/k) \cong H^2(\mathrm{Gal}(L/k), L^\times)$

Proof. We apply the inflation restriction exact sequence to the group G_k with normal subgroup G_L, both acting on the Galois-module $\overline{k}^\times$. We have that $\overline{k} = \overline{L}$, $(\overline{k}^\times)^{G_L} = \overline{L}^\times$, $G_k/G_L = \mathrm{Gal}(L/k)$, and Hilbert's theorem 90 (particularly applied to $H^1(G_L, \overline{L}^\times)$) gives

$$0 \to H^2(\mathrm{Gal}(L/k), L^\times) \to H^2(G_k, \overline{k}^\times) \to H^2(G_L, \overline{L}^\times)$$

Using the cohomological definition of the Brauer group of a field, the above sequence becomes

$$0 \to H^2(\mathrm{Gal}(L/k), L^\times) \to \mathrm{Br}\ k \to \mathrm{Br}\ L$$

hence ker (Br $k \to$ Br L) $\cong$ Br $(L/k) \cong H^2(\mathrm{Gal}(L/k), L^\times)$. □

The role that quaternion algebras play in the Brauer group will be our focus from here on out.

Proposition 5. [GS17, Lemma 1.5.2] *Given elements $a, b, b' \in k^\times$ we have an isomorphism*

$$(a,b) \otimes_k (a,b') \cong (a, bb') \otimes M_2(k)$$

Corollary 1. *For the quaternion algebra (a,b) we have the isomorphism $(a,b) \otimes_k (a,b) \cong M_4(k)$.*

Proof. Applying the above proposition with $b = b'$ we have

$$(a,b) \otimes (a,b) \cong (a, b^2) \otimes M_2(k) \cong M_2(k) \otimes M_2(k) \cong M_4(k)$$

□

The corollary implies that the Brauer class of any quaternion algebra is 2-torsion in the Brauer group. In fact, it is a theorem of Merkurjev [Voi19, Theorem 8.3.5] that all the 2-torsion in the Brauer group is accounted for by classes of quaternion algebras.

Definition 6. Let L/k be a cyclic extension of degree n with σ a fixed generator of $\mathrm{Gal}(L/k)$ and let $b \in k^\times$. We define the *cyclic algebra* (σ, b) to be the L-vector space

$$\frac{L \oplus Ly \oplus Ly^2 \oplus \cdots \oplus Ly^{n-1}}{y^n = b,\ \sigma(\alpha)y = y\alpha\ \forall \alpha \in L}$$

Remark 6. With a bit of work, one can show that (σ, b) is a central simple algebra defined over k and split by L [GS17, section 2.5]. Moreover, this generalizes the construction of quaternion algebras. In general, if k contains the nth roots of unity, then by Kummer theory, any cyclic extesion L of degree n is of the form $L = k(\sqrt[n]{a})$ for some $a \in k^\times$. Letting σ be a generator of $\mathrm{Gal}(L/k)$, we denote the cyclic algebra (σ, b) by $(a,b)_n$. For any quadratic extension $k(\sqrt{a})$ with $\sigma(\sqrt{a}) = -\sqrt{a}$, we have $(\sigma, b) \cong (a,b)$ thus recovering the generalized the quaternion algebras defined earlier. This shows that quaternion algebras are indeed central simple algebras.

Given any cyclic extension L/k we also have a nice description of the relative Brauer group

Proposition 6. [Gui18, Cor 7.19] *We have an explicit isomorphism*

$$\frac{k^\times}{N_{L/k}(L^\times)} \xrightarrow{\sim} Br(L/k), \quad b \mapsto (\sigma, b)$$

In particular, we can see that (σ, b) is trivial in Br k if and only if b is a norm from $L^\times$.

Corollary 2. $\mathrm{Br}\,\mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$

Proof. We can begin by observing that Br $\mathbb{R}$ = Br $(\mathbb{C}/\mathbb{R})$. Moreover, $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}_{>0}$ so Proposition 6 implies that

$$\mathrm{Br}\ \mathbb{R} = \mathbb{R}/\mathbb{R}_{>0} \cong \mathbb{Z}/2\mathbb{Z}$$

To find an explicit representative of the one nontrivial Brauer classe, it suffices to take any quaternion algebra generated by two elements of $\mathbb{R}_{<0}$ since they are not norms from $\mathbb{C}^\times$. Taking the quaternion algebra $(-1,-1)$, otherwise known as Hamilton's quaternions, will do. □

The situation over a local field k_v paints a much simpler picture. In particular, one can see that every Brauer class can be represented by a cyclic algebra.

2.3. **The Brauer Group of a Local Field.** Let k_v denote a non-archimedean local field. As defined in [Mil13, section 2.2], there is an isomorphism

$$\operatorname{inv}_v\colon \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$$

known as the Hasse invariant. The invariant map will be of central importance in computing the Brauer-Manin set. Here we exhibit some nice properties of this map, when applied to cyclic algebras.

Let L/k_v be an unramified (cyclic) extension and let $\sigma \in \operatorname{Gal}(L/k_v)$ inducing the Frobenius map on the residue field, then by [Mil13, Chap IV, Ex. 4.2 and Prop 4.3] we have

$$\operatorname{inv}_v((\sigma, b)) = \frac{v(b)}{[L : k_v]} \in \mathbb{Q}/\mathbb{Z}$$

Remark 7. If we restrict the domain of the invariant map to Br $(k_v)[2]$, namely the quaternion algebras, then the isomorphism shows that there exists a unique non-trivial 2-torsion class. In other words, $\operatorname{inv}_v|_{\operatorname{Br}(k_v)[2]}\colon \operatorname{Br}(k_v)[2] \xrightarrow{\sim} \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

2.4. **The Brauer Group of a Global Field.** Let k be a global field and let Ω_k denote the set of places of k. The fundamental exact sequence of global class field theory completely characterizes the Brauer group of any global field.

$$0 \to \operatorname{Br}\ k \to \bigoplus_{v\in\Omega_k} \operatorname{Br}\ k_v \xrightarrow{\sum_v \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0 \tag{2.1}$$

The Brauer group of k is identified with a subgroup of the direct sum $\bigoplus_{v\in\Omega_k} \operatorname{Br} k_v$ and the inclusion map is given by tensoring any central simple algebra over k with each completion. The Brauer group of each completion at nonarchimedian places is isomorphic to $\mathbb{Q}/\mathbb{Z}$ and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ at the archimedian place. The Brauer group of k is then the kernel of the sum of local invariants.

It is worth noting that not only is this group infinite, but even the 2-torsion is infinite. In fact, even the relative Brauer group Br $(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is infinite!

3. The Brauer-Manin Obstruction

Throughout this section, let X be a nice variety over a global field k of characteristic zero.

3.1. **The Brauer Group of a Scheme.** The set $X(\mathbb{A}_k)^{\operatorname{Br}}$ is defined via the Brauer group of X, which is defined as follows.

Definition 7. The Brauer Group of a Scheme Given a scheme X we define

$$\operatorname{Br} X \coloneqq H^2_{\text{ét}}(X, \mathbb{G}_m)$$

By functoriality of cohomology, Br is a functor from the category of schemes to the category of abelian groups. However, in contrast to fields, Br is **contra**variant for schemes.

Remark 8. Note that this definition generalizes the notion of the Brauer group of a field, and given a field k, we have $\operatorname{Br} k = H^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$. Furthermore, when X is nice, Br X is torsion [Mil80, Example III.2.22], as is the case for fields.

Theorem 4. [Gro68b, Corollarie 7.5] *If X is a smooth projective surface, then* $\operatorname{Br} X$ *depends only on the birational class of X.*

Corollary 3. *Let X be a nice geometrically rational surface over a field k. Then* $\operatorname{Br} \overline{X} = 0$

Proof. From Theorem 4, we have an isomorphism $\operatorname{Br} \overline{X} \cong \operatorname{Br} \mathbb{P}^n_{\overline{k}}$. Moreover, the induced map $\operatorname{Br} k \to \operatorname{Br} \mathbb{P}^n_k$ coming from the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ is an isomorphism, see [Poo10, Proposition 6.9.9]. Combining this with Lemma 1, we get the string of isomorphisms $\operatorname{Br} \overline{X} \cong \operatorname{Br} \mathbb{P}^n_{\overline{k}} \cong \operatorname{Br} \overline{k} = 0$. □

Given a morphism $X \to \operatorname{Spec} k$ we obtain a map on Brauer classes $\operatorname{Br} k \to \operatorname{Br} X$ by pulling back. Let X be a variety over a field and define

$$\operatorname{Br}_0\ X \coloneqq \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) \quad \text{and} \quad \operatorname{Br}_1\ X \coloneqq \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$$

One can show the inclusion $\operatorname{Br}_0 X \subset \operatorname{Br}_1 X \subset \operatorname{Br} X$. Elements of $\operatorname{Br}_0 X$ are called *constant* and elements of $\operatorname{Br}_1 X$ are called *algebraic*.

Proposition 7. *If X is geometrically rational then* $\operatorname{Br}_1 X = \operatorname{Br} X$. *If X is over a global field k and $X(\mathbb{A}_k) \neq \emptyset$ then* $\operatorname{Br} k = \operatorname{Br}_0 X$.

Proof. If $\overline{X}$ is rational, then by Corollary 3, $\operatorname{Br}_1 X = \operatorname{Br} X$. To show that the natural map $\operatorname{Br} k \to \operatorname{Br} X$ is injective, the existence of an adélic point gives a map $P_v : \operatorname{Spec} k_v \to X_{k_v}$ as in the following diagram

$$\begin{array}{ccc} X_{k_v} & \longrightarrow & X \\ {\scriptstyle P_v}\uparrow\downarrow{\scriptstyle \pi_v} & & \downarrow \\ \operatorname{Spec} k_v & \longrightarrow & \operatorname{Spec} k \end{array}$$

By functoriality of Brauer groups we have

$$\begin{array}{ccc} \operatorname{Br} X & \longrightarrow & \operatorname{Br} X_{k_v} \\ \uparrow & & {\scriptstyle \pi_v^*}\uparrow\downarrow{\scriptstyle P_v^*} \\ \operatorname{Br} k & \longrightarrow & \operatorname{Br} k_v \end{array}$$

We can see that P_v splits the base change of the structure morphism of X for every $v \in \Omega_k$ hence the natural maps $\pi_v^* : \operatorname{Br} k_v \to \operatorname{Br} X_{k_v}$ split for every $v \in \Omega_k$. Combining this with (2.1), it follows that the induced map $\operatorname{Br} k \to \operatorname{Br} X$ coming from the structure map of X is injective. □

Given a commutative ring R, we set $\operatorname{Br} R \coloneqq \operatorname{Br}(\operatorname{Spec} R)$.

Lemma 2. [Mil80, III.3.11(a)] *Let R be a non-archimedian local ring with residue field $\mathbb{F}$. The quotient map $R \to k$ induces an isomorphism* $\operatorname{Br} R \xrightarrow{\sim} \operatorname{Br} \mathbb{F}$

Corollary 4. *Let k be a complete valued field with valuation denoted by v and valuation ring $\mathcal{O}_v$. Then* $\operatorname{Br} \mathcal{O}_v = 0$.

Proof. By Lemma 2, $\mathcal{O}_v \to \mathbb{F}_v$ induces the isomorphism $\operatorname{Br} \mathbb{F}_v \cong \operatorname{Br} \mathcal{O}_v$. For any non-archimedian local field, $\mathbb{F}_v$ is finite. Since every finite division algebra is a field, the only central simple algebras over $\mathbb{F}_v$ are matrix algebras. Hence $\operatorname{Br} \mathbb{F}_v = 0$. □

3.2. **The Brauer-Manin Set.** Let Ω_k denote the set of places of k. If $v \in \Omega_k$, write k_v for the completion of k at v and $\mathcal{O}_v$ for the valuation ring in k_v. Let $\mathbb{A}_k$ be the adéle ring of k, that is, $\mathbb{A}_k$ is the restricted product $\mathbb{A}_k = \prod'_{v\in\Omega_k}(k_v, \mathcal{O}_v)$. This is a subring of the product $\prod_{v\in\Omega_k} k_v$ containing tuples (P_v) for which there exists a finite set of places S such that $P_v \in \mathcal{O}_v$ for all $v \notin S$.

Let $X(\mathbb{A}_k)$ denote the set of adélic points of X. A priori, $X(\mathbb{A}_k)$ is a subset of $X(\prod_{v\in\Omega_k} k_v)$ but one can show ([Poo10], Exercise 3.4) that if X is proper, then $X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$.

For any $P_v \in X(k_v)$ and any $\mathcal{A} \in \operatorname{Br} X$, we can pullback $\mathcal{A}$ along $P_v\colon \operatorname{Spec} k_v \to X$ and obtain an element of Br k_v. We denote this element $\mathcal{A}(P_v)$ and regard it as the image of P_v under the map $\operatorname{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v$, which we call the evaluation map. A priori, this gives a map $\operatorname{ev}_{\mathcal{A}}\colon X(\mathbb{A}_k) \to \prod_{v\in\Omega_k} \operatorname{Br} k_v$.

Given a point $(P_v) \in X(\mathbb{A}_k)$, for some finite set of places $S \subset \Omega_k$, there exists a finite type $\mathcal{O}_{k,S}$ scheme $\mathcal{X}$ equipped with a map $X \hookrightarrow \mathcal{X}$, such that for fixed $\mathcal{A} \in \operatorname{Br} X$, one can find $\widetilde{\mathcal{A}} \in \operatorname{Br} \mathcal{X}$ with $\widetilde{\mathcal{A}}$ pulling back to $\mathcal{A}$ under $\operatorname{Br} \mathcal{X} \to \operatorname{Br} X$, [Poo10, Corollary 6.6.11]. In particular, for a k_v point P_v, we have $P_v(\operatorname{Spec} k_v) \in \operatorname{Spec} A \subset X$ and since X is finite type, $A = k_v[x_1, ..., x_n]/(f_1, ..., f_r)$. We take S to be the (finite) set of valuations that are negative when applied to the finitely many coefficients of the f_i. For all $v \notin S$, we have $P_v \in X(\mathcal{O}_v)$. This implies that for all but finitely many v, $\operatorname{ev}_{\mathcal{A}}(P_v) \in \operatorname{Br} \mathcal{O}_v$. By Corollary 4, $\operatorname{Br} \mathcal{O}_v = 0$ so $\operatorname{ev}_{\mathcal{A}}(P_v) = 0$ for almost all v. Therefore, $\operatorname{ev}_{\mathcal{A}}$ gives a map $X(\mathbb{A}_k) \to \oplus_{v\in\Omega_k} \operatorname{Br} k_v$.

Composing this with the sum of local invariants, $\operatorname{inv}_v : \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$, we obtain a well-defined map $X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ given by

$$(P_v) \mapsto \sum_{v\in\Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v))$$

Before defining the Brauer-Manin set, we cite one lemma that will be of use in the coming results.

Lemma 3. [Vir10, Lemma 3.3.2] *Let k_v be a local field. For any $\mathcal{A} \in \operatorname{Br} X$*

$$\operatorname{ev}_{\mathcal{A}}\colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$$

is continuous for the discrete topology on $\mathbb{Q}/\mathbb{Z}$

Definition 8. (**The Brauer-Manin Set**) Given $\mathcal{A} \in \operatorname{Br} X$ let

$$X(\mathbb{A}_k)^{\mathcal{A}} = \left\{ (P_v) \in X(\mathbb{A}_k)\colon \sum_{v\in\Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 0 \right\}$$

We call

$$X(\mathbb{A}_k)^{\operatorname{Br}} = \bigcap_{\mathcal{A}\in \operatorname{Br} X} X(\mathbb{A}_k)^{\mathcal{A}}$$

the *Brauer Manin set.*

Lemma 3 shows that $X(\mathbb{A}_k)^{\mathcal{A}}$ is a closed subset of $X(\mathbb{A}_k)$.

Proposition 8. $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$

Proof. For $\mathcal{A} \in \operatorname{Br} X$, we obtain a commuting diagram

$$\begin{array}{ccccccccc} & & X(k) & \longrightarrow & X(\mathbb{A}_k) & & & & \\ & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & & & \\ 0 & \longrightarrow & \operatorname{Br} k & \xrightarrow{\phi} & \bigoplus_{v\in\Omega_k} \operatorname{Br} k_v & \xrightarrow{\sum_v \mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where the bottom row is the usual exact sequence from global class field theory. It is immediate from the definition of $X(\mathbb{A}_k)^{\mathrm{Br}}$ that $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. To see the other inclusion, take $P \in X(k)$ and its image $(P_v) \in X(\mathbb{A}_k)$. Commutativity of the diagram implies that $\phi(\mathrm{ev}_{\mathcal{A}}(P)) = \mathrm{ev}_{\mathcal{A}}(P_v)$. Moreover, exactness of the bottom row then gives $\sum_v \mathrm{inv}_v(\phi(\mathrm{ev}_{\mathcal{A}}(P))) = \sum_v \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0$ hence $P \in X(\mathbb{A}_k)^{\mathcal{A}}$. □

We say there is a **Brauer-Manin obstruction to the Hasse principle** if $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$

Corollary 5. *For any* $\mathcal{A} \in \operatorname{Br}_0 X$, $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)$.

Proof. From Proposition 7 and the exactness of the bottom row of the diagram in Proposition 8, we have that for any $\mathcal{A} \in \operatorname{Br} k$

$$\mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0$$

for every v. □

Corollary 6. *To compute* $X(\mathbb{A}_k)^{\mathrm{Br}}$*, it is enough to compute the intersection over a set of representatives of* $\operatorname{Br} X / \operatorname{Br}_0 X$.

Proof. This is immediate from the above Corollary. □

3.3. **The Hochschild-Serre Spectral Sequence in Étale Cohomology.** Let X be a nice variety over a global field k that is everywhere locally solvable. If X is geometrically rational, then the Hochschild-Serre spectral sequence in étale cohomology gives a tool for computing the group $\operatorname{Br} X / \operatorname{Br}_0 X$.

Proposition 9. *Let* k *be a global field and* X/k *a nice geometrically rational variety with* $X(\mathbb{A}_k) \neq \emptyset$*. Then we obtain an exact sequence*

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{G_k} \to \operatorname{Br} k \to \operatorname{Br} X \to H^1(G_k, \operatorname{Pic} \overline{X}) \to H^3(k, \mathbb{G}_m)$$

Since k *is a global field,* $H^3(k, \mathbb{G}_m) = 0$ *and we have an isomorphism*

$$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X} \cong H^1(G_k, \operatorname{Pic} \overline{X})$$

Proof. Let L/k be a finite extension with $G = \operatorname{Gal}(L/k)$. The Hochschild-Serre sprectral sequence

$$E_2^{p,q} := H^p(G, H^q_{\text{ét}}(X_L, \mathbb{G}_m)) \implies L^{p+q} := H^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

gives rise to the usual seven term exact sequence

$$0 \to E_2^{1,0} \to L^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker(L^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}$$

In our case, we get

$$0 \to H^1(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^0(G, H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^2(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m))$$
$$\to \ker(H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^0(G, H^2_{\text{ét}}(X_L, \mathbb{G}_m))) \to H^1(G, H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^3(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m))$$

One can show (see e.g. [Poo10, Proposition 6.6.1]) that

$$H^0_{\text{ét}}(X_L, \mathbb{G}_m)) \cong L^\times, \quad \text{and} \quad H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \cong \operatorname{Pic} X_L$$

which gives

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} X)^G \to H^2(G, L^\times) \to \ker(\operatorname{Br} X \to \operatorname{Br} X_L) \to H^1(G, \operatorname{Pic} X_L) \to H^3(G, L^\times)$$

In particular, for the extension $\overline{k}/k$ with Galois group G_k we get

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{G_k} \to \operatorname{Br} k \to \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X}) \to H^1(G_k, \operatorname{Pic} \overline{X}) \to H^3(G_k, \overline{k}^\times)$$

Furthermore, if k is a global field, then $H^3(G_k, \overline{k}^\times) = 0$, which is a result due to Tate, see [NSW08, 8.3.11(iv), 8.3.17]. Proposition 7 then gives rise to the short exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br} X \to H^1(G_k, \operatorname{Pic} \overline{X}) \to 0$$

yeilding the desired isomorphism □

Remark 9. When $X(\mathbb{A}_k) \neq \emptyset$, if H is a subgroup of G_k, then by looking at the first two terms of the above sequence, injectivity of the natural map $\operatorname{Br} k \to \operatorname{Br} X$ gives the isomorphism

$$\operatorname{Pic} X_{\overline{k}^H} \cong (\operatorname{Pic} \overline{X})^H$$

where $\overline{k}^H$ denotes the fixed field of H.

3.4. **Galois action on the Picard Group.** The Galois group $\operatorname{Gal}(\overline{k}/k)$ acts on $\operatorname{Pic} \overline{X}$ as follows. For $\sigma \in \operatorname{Gal}(\overline{k}/k)$ let $\tilde{\sigma} \in \operatorname{Aut}(\operatorname{Spec} \overline{k})$ be the corresponding morphism. By considering the base change of X to $\overline{k}$, the pullback of the morphism $\operatorname{id}_X \times \tilde{\sigma} \colon \overline{X} \to \overline{X}$ induces an automorphism $(\operatorname{id}_X \times \tilde{\sigma})^*$ of $\operatorname{Pic} \overline{X}$. This gives a group homomorphism

$$\operatorname{Gal}(\overline{k}/k) \to \operatorname{Aut}(\operatorname{Pic} \overline{X}), \quad \sigma \mapsto (\operatorname{id}_X \times \tilde{\sigma})^*$$

This action preserves the intersection pairing and thus exceptional curves to exceptional curves [Man74, Theorem 23.8].

If X has torsion free Picard group, we define the **splitting field** of X to be the smallest extension L of k in $\overline{k}$ for which the action of $\operatorname{Gal}(\overline{k}/L)$ on $\operatorname{Pic} \overline{X}$ is trivial.

Proposition 10. *Let X be a nice surface with torsion free geometric Picard group. Let L be the splitting field of X and let $\operatorname{Pic} \overline{X} \cong \mathbb{Z}^r$ for some $r > 0$. Then the inflation map $\inf \colon H^1(\operatorname{Gal}(L/k), (\operatorname{Pic} \overline{X})^H) \to H^1(G_k, \operatorname{Pic} \overline{X})$ is an isomorphism.*

Proof. Applying the inflation restriction exact sequence to the subgroup $H = \operatorname{Gal}(\overline{k}/L)$, we have $G_k/H = \operatorname{Gal}(L/k)$ and the first three terms of inflation restriction are

$$0 \to H^1(\operatorname{Gal}(L/k), (\operatorname{Pic} \overline{X})^H) \xrightarrow{\inf} H^1(G_k, \operatorname{Pic} \overline{X}) \xrightarrow{\text{res}} H^1(\operatorname{Gal}(\overline{k}/L), \operatorname{Pic} \overline{X})^{\operatorname{Gal}(L/k)}$$

Since $\mathrm{Gal}(\overline{k}/L)$ is a limit of finite Galois groups, each of which acts trivially on the free abelian group $\mathbb{Z}^r$, it follows that $H^1(\mathrm{Gal}\,\overline{k}/L, \mathrm{Pic}\,\overline{X}) = 0$ because it is a limit of the groups $\mathrm{Hom}(H, \mathbb{Z}^r)$ which are all trivial, since H is finite. The result now follows. □

If we further assume that X has adélic points, then Remark 9 gives

$$H^1(\mathrm{Gal}(L/k), \mathrm{Pic}\,X_L) \cong H^1(G_k, \mathrm{Pic}\,\overline{X})$$

Moreover, if we have irreducible curves C on X cut out by equations with coefficients in L, then an element $\sigma \in \mathrm{Gal}(L/k)$ acts on C by applying σ to each coefficient.

4. The Geometry of Châtelet Surfaces

The goal of this section is to define Châtelet surfaces and then prove that over $\overline{k}$, a Châtelet surface is the blow-up of a Hirzebruch surface at four points. In other words, that Châtelet surfaces are geometrically rational. We begin with some background on ruled surfaces.

4.1. **Background.** Let X be a nice surface, let C, D be divisors on X that intersect properly and take $C.D = \#(C \cap D) = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C)$ to be the usual intersection pairing on Div $X \times$ Div X [Har77, V.3]. For a nice variety, the Picard group $\mathrm{Pic}\,X$ coincides with the group of Weil divisors modulo linear equivalence.

Definition 9. (**Ruled Surface**) A ruled surface is a surface X, together with a surjective morphism $\pi\colon X \to C$ to a nonsingular curve C, such that
1) for every point $y \in C$, the fiber X_y is isomorphic to $\mathbb{P}^1$
2) π admits a section, $\sigma\colon C \to X$.

Proposition 11. [Har77, Proposition V.2.2]
If $\pi\colon X \to C$ is a ruled surface, then there exists a locally free sheaf $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C. Conversely, every such $\mathbb{P}(\mathcal{E})$ is a ruled surface over C. Additionally, $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$ if and only if there exists a line bundle $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$.

Definition 10. (Hirzebruch Surface) A Hirzebruch surface $\mathbb{F}_n$ is a ruled surface associated to the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-n)$ for $n \geq 0$.

Definition 11. (Blowing up a variety at a point) Let X be a variety and P a closed point of X. The blow-up of X at P is a variety $\widetilde{X}$ equipped with a morphism $\pi\colon \widetilde{X} \to X$ such that π induces an isomorphism of $\widetilde{X} - \pi^{-1}(P)$ to $X - P$. The preimage of P under π is called the *exceptional divisor* of the blow-up, and we denote it by E. Moreover, given any divisor D (containing P) on X, we define the *strict transform* of D, denoted $\widetilde{D}$, to be the closure of $\pi^{-1}(D - P)$.

Proposition 12. [Har77, Proposition V.3.2] *The intersection theory on $\widetilde{X}$ is defined by the following rules:*

- *If $C, D \in$ Pic X, then $(\pi^*C).(\pi^*D) = C.D$*
- *If $C \in$ Pic X, then $(\pi^*C).E = 0$*
- *$E^2 = -1$*

We refer to any curve on X of self-intersection $-n$ as a $(-n)$-curve.

Proposition 13. [Har77, Proposition V.3.6] *Let C be an effective divisor on X, let P be a point of multiplicty r on C, and let $\pi : \widetilde{X} \to X$ be the blow-up of X at P. Then*

$$\pi^* C = \widetilde{C} + rE$$

Theorem 5. ***(Castelnuovo)***[Har77, Theorem V.5.7] *If Y is a (-1)-curve on a smooth projective surface X, with $Y \cong \mathbb{P}^1$, then there exists a nonsingular projective surface X_0, a point $P \in X_0$, and a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathrm{Bl}_P X_0 \\ \downarrow f & \swarrow & \\ X_0 & & \end{array}$$

This result allows us to blow-down all (-1)-curves on X and obtain the "simplest" birational model of X. As a result of Theorem 3, we will want to know when a given surface is rational. Due to the work of Castelnuovo, there is an especially nice criterion to determine this. Define $q(S) \coloneqq h^1(S, \mathcal{O}_S)$ and for $n \geq 1$, $P_n(S) \coloneqq h^0(S, K_S^{\otimes n})$ where K_S denotes the canonical divisor on S. We then have

Theorem 6. ***(Castelnuovo's Rationality Criterion)***[Bea78, Theorem V.1] *Let S be a surface. If $q(S) = P_2(S) = 0$ then S is rational.*

Proposition 14. [Bea78, Proposition III.21] *Let S be a ruled surface over C and let g be the genus of C.Then*

$$q(S) = g \quad and \quad P_n(S) = 0 \quad \forall n \geq 2$$

Corollary 7. *$\mathbb{F}_n$ is rational.*

For more on rational surfaces see [Bea78].

We finish out our preliminary results with two important facts concerning Picard groups.

Proposition 15. ***(Picard Group of a ruled surface)***[Har77, Prop V.2.3]
Let $\pi : X \to C$ be a ruled surface, let $\sigma(C) \cong C_0 \subset X$ be a section, and F be a fiber. Then

$$\text{Pic } X \cong \mathbb{Z} \oplus \pi^* \text{Pic } C$$

with $\mathbb{Z}$ generated by C_0. Additionally, C_0 and F satisfy $C_0.F = 1, F^2 = 0$.

Proposition 16. ***(Picard Group of a blow up)***[Har77, Prop V.3.2]
Given the natural map $\pi : \widetilde{X} \to X$, we have maps $\pi^* : Pic\ X \to Pic\ \widetilde{X}$ and $\mathbb{Z} \to Pic\ \widetilde{X}$ defined by $1 \mapsto 1 \cdot E$ which give rise to the isomorphism

$$Pic\ \widetilde{X} \cong Pic\ X \oplus \mathbb{Z}$$

4.2. **Châtelet Surfaces.** Let $P(\lambda) \in k[\lambda]$ be a separable polynomial of degree 4 and let $a \in k^\times$. Take

$$X_1 \coloneqq \text{Proj } \frac{k[\lambda][y, z, t]}{(y^2 - az^2 - P(\lambda)t^2)} \subset \mathbb{P}^2_{\mathbb{A}^1_k}$$

with coordinates $(y : z : t, \lambda)$ and

$$X_2 \coloneqq \text{Proj } \frac{k[\mu][Y, Z, T]}{(Y^2 - aZ^2 - Q(\mu)T^2)} \subset \mathbb{P}^2_{\mathbb{A}^1_k}$$

with coordinates $(Y : Z : T, \mu)$ and

$$Q(\mu) = \mu^4 P\Big(\frac{1}{\mu}\Big)$$

Let X be the surface obtained by gluing X_1 and X_2 via the isomorphism

$$X_1 - \{\lambda = 0\} \xrightarrow{\sim} X_2 - \{\mu = 0\}$$
$$(y : z : t, \lambda) \mapsto (Y : Z : \mu^2 T, 1/\mu)$$

which is indeed an isomorphism because

$$(y^2 - az^2 - P(\lambda)t^2) \mapsto (Y^2 - aZ^2 - P(1/\mu)\mu^4 T^2) = (Y^2 - aZ^2 - Q(\mu)T^2)$$

Definition 12. We call X the **Châtelet surface** given by $y^2 - az^2 = P(\lambda)$.

Observe that X comes with a morphism to $\pi\colon X \to \mathbb{P}^1_k$ obtained by gluing the projections $X_1 \to \mathbb{A}^1_k$ and $X_2 \to \mathbb{A}^1_k$ given by $(y : z : t, \lambda) \mapsto \lambda$ and $(Y : Z : T, \mu) \mapsto \mu$, respectively.

We note here that the original surfaces studied by Châtelet [Châ59] took $P(\lambda)$ to be degree 3 or 4 but we can obtain the degree 4 case from the degree 3 case by homogenizing, and if necessary, making a linear change of variables that shifts a root away from the point at infinity.

Proposition 17. *X is smooth, projective, and geometrically integral.*

Proof. To show smoothness, we can use the Jacobian criterion on an open cover of X. We can take as our open cover the 6 open affines given by $y \neq 0, z \neq 0$, and $t \neq 0$ on the $\operatorname{Spec} k[\lambda]$ patch, and $Y \neq 0, Z \neq 0$, and $T \neq 0$ on the $\operatorname{Spec} k[\mu]$ patch. It is easy to see that the Jacobian has rank 1 on each patch, and the only non-trivial computation happens where $t \neq 0$. The Jacobian here is given by

$$\begin{bmatrix} 2y & -2az & \frac{\partial P}{\partial \lambda} \end{bmatrix}$$

This has rank 1 at all points of X because separability of P implies that $\frac{\partial P}{\partial \lambda}$ and P are coprime.

To see that X is projective, we show it is proper. Since properness is local on the base, it is clear that π is a projective morphism on the standard affine cover of $\mathbb{P}^1_k$, hence π is proper. Since smooth proper **surfaces** are projective, X is projective. Note, Hironaka's example shows that smooth proper schemes are not projective in general, but when we restrict to the case of surfaces, it is in fact true [Har70, II.4.2].

Finally, to see that it is geometrically integral, note that smoothness implies regularity and regular local rings are always reduced, so it suffices to check that $\overline{X}$ is irreducible. It is enough to check it on an open cover. We first observe that the 6 standard affines intersect non-trivially, and on the open affine $t \neq 0, \mu \neq 0$, we consider the ring $\frac{k[\lambda][y,z,t]}{(y^2 - az^2 - P(\lambda))}$. Since a is a square in $\overline{k}$, consider the prime ideal $\mathfrak{p} = (y - \sqrt{a}z)$. By generalized Eisenstein, $y^2 - az^2 - P(\lambda)$ is irreducible so long as $P(\lambda)$ is not identically zero. This implies that $\frac{\overline{k}[\lambda][y,z,t]}{(y^2 - az^2 - P(\lambda))}$ is an integral domain. Verifying irreducibility on the other open affines follows similarly. ∎

As the ground field changes so does the Galois action on $\operatorname{Pic} \overline{X}$. Let $\sqrt{a}$ denote a fixed square root of a. When we base change X to $k(\sqrt{a})$ we obtain two sections C_0 and $\widetilde{C}_0$ which extend the sections $\lambda \mapsto (\sqrt{a} : 1 : 0, \lambda)$ and $\lambda \mapsto (-\sqrt{a} : 1 : 0, \lambda)$ respectively of the

projection $X_{1_{k(\sqrt{a})}} \to \mathbb{A}^1_{k(\sqrt{a})}$
Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be the roots of P in $\overline{k}$ so that we have a factorization of P given by

$$P(\lambda) = c\prod_{i=1}^{4}(\lambda - \lambda_i)$$

Denote the base change of our structure morphism by $\overline{\pi} : \overline{X} \to \mathbb{P}^1_{\overline{k}}$. All fibers above points of $\mathbb{P}^1_{\overline{k}}$ are smooth conics except for the 4 fibers above the roots λ_i on the open set $\overline{X}_1$, which degenerate to the union of two intersecting lines. We denote these subschemes of $\overline{X}_1$ by

$$E_i = \{\lambda = \lambda_i, y - \sqrt{a}z\} \quad \text{and} \quad \widetilde{E}_i = \{\lambda = \lambda_i, y + \sqrt{a}z\}$$

and both E_i and $\widetilde{E}_i$ are isomorphic to $\mathbb{P}^1_{\overline{k}}$. Finally, we let F denote the smooth conic over ∞ $(\mu = 0)$.

Proposition 18.

$$\mathrm{Div}_{\overline{X}}(\lambda - \lambda_i) = E_i + \widetilde{E}_i - F \tag{4.1}$$

$$\mathrm{Div}_{\overline{X}}\left(\frac{y - \sqrt{a}z}{t}\right) = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0 - 2F \tag{4.2}$$

Proof. To compute $\mathrm{Div}_{\overline{X}}(\lambda-\lambda_i)$ we observe that over Spec $k[\lambda]$ it suffices to compute $V(\lambda-\lambda_i)$ to find the zeroes, this gives $E_i + \widetilde{E_i}$. Over Spec $k[\mu]$ we apply the transition function $\lambda \mapsto \frac{1}{\mu}$ to obtain $\frac{1-\mu\lambda_i}{\mu}$. This has a pole at $\mu = 0$ which gives the smooth fiber F hence

$$\mathrm{Div}_{\overline{X}}(\lambda - \lambda_i) = E_i + \widetilde{E}_i - F$$

To carry out the remaining computation, we must compute valuations of the rational function $\frac{y-\sqrt{a}z}{t}$ over each affine patch, of which there are 6 . To do this we must find the codimension 1 primes **containing** $y - \sqrt{a}z$ or t. By localizing the appropriate ring at these primes, we obtain a DVR that has non-trivial valuation when applied to the functions in question. When localizing at the codimension 1 primes not containing the above functions, $y - \sqrt{a}z$ and t will become units in the localization, hence we safely ignore those.

To find these primes, we first let $A = \frac{\overline{k}[\lambda][y,z,t]}{(y^2-az^2-P(\lambda)t^2)}$ and then use the fact that codimension 1 primes containing $y - \sqrt{a}z$ or t, respectively correspond to codimension 0 (minimal) primes in $A/(y - \sqrt{a}z)$ and A/t, respectively. After finding these primes $\mathfrak{p}$, we find uniformizers in $A_{\mathfrak{p}}$ by using the relation $y^2 - az^2 = P(\lambda)t^2$, and then compute valuations. We give divisors according to the equations that cut them out:

- $E_i = \{\lambda = \lambda_i, y - \sqrt{a}z = 0\}$
- $\widetilde{E}_i = \{\lambda = \lambda_i, y + \sqrt{a}z = 0\}$
- $C_0 = \{y - \sqrt{a}z = 0, t = 0\}$
- $\widetilde{C}_0 = \{y + \sqrt{a}z = 0, t = 0\}$
- $F = \{\mu = 0\}$

We can now begin the computation:

- $(z \neq 0, \mu \neq 0)$. We first find codimension 1 primes containing (t), hence look for minimal primes in $A/(t)$.

$$A/(t) = \frac{k[\lambda][y]}{(y^2 - a)} = \frac{\overline{k}[\lambda, y]}{(y + \sqrt{a})} \times \frac{\overline{k}[\lambda, y]}{(y - \sqrt{a})} = \overline{k}[\lambda]^2$$

The minimal primes are clearly $(y \pm \sqrt{a})$ and the corresponding codimension 1 primes in A are simply obtained by lifting (i.e. adding t as a generator) thus yielding the primes $\mathfrak{p}_1 = (y + \sqrt{a}, t)$ and $\mathfrak{p}_2 = (y - \sqrt{a}, t)$

In the local ring $A_{\mathfrak{p}_1}$, we can write the relation $y^2 - az^2 = P(\lambda)t^2$ as

$$y + \sqrt{a} = t^2 \Big(\frac{P(\lambda)}{y - \sqrt{a}} \Big)$$

Since $\frac{P(\lambda)}{y - \sqrt{a}}$ is a unit we have $y + \sqrt{a} \subset (t)$, hence t generates $\mathfrak{p}_1$ and $v_{\mathfrak{p}_1}(t) = 1$. An almost identical calculation shows that t is also a uniformizer in $A_{\mathfrak{p}_2}$. As a result, we get t vanishing to order 1.

Finding the codimension 1 primes containing $y - \sqrt{a}$ on the same patch, we look at

$$A/(y - \sqrt{a}) = \frac{\overline{k}[\lambda][y, t]}{(y^2 - a - P(\lambda)t^2, y - \sqrt{a})} = \frac{\overline{k}[\lambda, t]}{(P(\lambda)t^2)} = \prod_{i=1}^{4} \frac{\overline{k}[\lambda, t]}{(\lambda - \lambda_i)} \times \frac{\overline{k}[\lambda, t]}{(t^2)}$$

and after lifting, we get minimal primes $\mathfrak{q}_i = (\lambda - \lambda_i, y - \sqrt{a})$ and $\mathfrak{p}_2 = (t, y - \sqrt{a})$. In the local ring $A_{\mathfrak{q}_i}$, the relation

$$(y - \sqrt{a})(y + \sqrt{a}) = \Big(\prod_{j=1}^{4} (\lambda - \lambda_i) \Big) t^2$$

gives us the two expressions

$$y - \sqrt{a} = \frac{\Big(\prod_{j=1}^{4} (\lambda - \lambda_i) \Big) t^2}{(y + \sqrt{a})} \quad \text{and} \quad \lambda - \lambda_i = (y - \sqrt{a}) \Big(\frac{y + \sqrt{a}}{\prod_{i \neq j} (\lambda - \lambda_i) t^2} \Big)$$

Thus both $\lambda - \lambda_i$ and $y - \sqrt{a}$ are uniformizers and at all codimension 1 points $\mathfrak{p}_i$, $\lambda - \lambda_i$ and $y - \sqrt{a}$ vanish to order 1. Since $V(\mathfrak{q}_i)$ is precisely E_i, we know we will have $\sum E_i$ in our computation.

Localizing at $\mathfrak{q} = (y - \sqrt{a}, t)$ we have $y - \sqrt{a} = (\text{unit})t^2$ thus t is a uniformizer and by applying $v_{\mathfrak{q}}$ to the previous equation we get $v_{\mathfrak{q}}(y - \sqrt{a}) = v_{\mathfrak{q}}(t^2) = 2$. We complete the computation on the $z \neq 0$ patch by observing that at the prime $\mathfrak{q}$, the divisor corresponding to $(y - \sqrt{a}, t)$ vanishes to order 2. This divisor is C_0. Directly computing the valuation of the rational function $\frac{y - \sqrt{a}}{t}$ in these localizations, we obtain

$$\mathrm{Div}_{\overline{X}_{z \neq 0}} \Big(\frac{y - \sqrt{a}}{t} \Big) = E_1 + E_2 + E_3 + E_4 + 2C_0 - (C_0 + \widetilde{C}_0) = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0$$

- $(y \neq 0, \mu \neq 0)$ We make a simplification by noting that the transition function from the $z \neq 0$ patch to the $y \neq 0$ patch is multiplication by $\frac{z}{y}$. On the $z \neq 0$ patch, we are given the function $\frac{\frac{y}{z}-\sqrt{a}}{t}$ and moving to the patch where $y \neq 0$ we get
$$\frac{\frac{z}{y}(\frac{y}{z} - \sqrt{a})}{t} = \frac{1 - z\sqrt{a}}{t}$$
By observing how we determined the order of vanishing of t in the local rings $A_{\mathfrak{p}_1}$ and $A_{\mathfrak{p}_2}$ and the order of vanishing of $y - \sqrt{a}$ in $A_{\mathfrak{p}_i}$ and $A_{\mathfrak{q}}$, we see that applying the transition function, while changing the ideals, does not give new divisors in our computation. As a result, any divisor coming from the computation where $y \neq 0$ will be double counted, hence there is nothing more to compute.
- $(t \neq 0, \mu \neq 0)$ This patch is defined by $\mathrm{Spec}(A_t)_0$ so t is a unit and there are no (codimension 1) primes containing t. Moreover, to find primes containing $y - \sqrt{a}z$ we look at $A/(y - \sqrt{a}z)$ which is
$$\frac{\overline{k}[\lambda][y, z, t]}{(y^2 - az^2 - P(\lambda), y - \sqrt{a}z)} = \frac{\overline{k}[\lambda]}{P(\lambda)} = \prod \frac{\overline{k}[\lambda]}{(\lambda - \lambda_i)} = \overline{k}^{\times 4}$$
There are 4 minimal primes, namely $(\lambda - \lambda_i)$ but this gives no new information.

- We now move to the other standard open affine of $\mathbb{P}^1_{\overline{k}}$. Write $Q(\mu) = \sum_{i=1}^4 a_i\mu^i$ and observe that under the map $(y : z : t, \lambda) \mapsto (Y : Z : \mu^2 T, 1/\mu)$ our rational function is sent to
$$\frac{y - \sqrt{a}z}{t} \mapsto \frac{Y - \sqrt{a}Z}{\mu^2 T}$$
When $Y, Z \neq 0$ the divisor computation is identical to what we did when $\mu \neq 0$. Moreover, we have looked at all possibilities when $\mu \neq 0$ so it remains to check what happens when $\mu = 0$ on either of the patches $Y \neq 0$ or $Z \neq 0$.
- $(Y \neq 0, \lambda \neq 0)$ Let $B = \frac{\overline{k}[\mu][Z,T]}{(1-aZ^2-Q(\mu)T^2)}$ and let a_0 the constant term of $Q(\mu)$. Note that a_0 is the leading term of $P(\lambda)$ so must be nonzero. On the locus where $\mu = 0$ we get
$$B/\mu = \frac{\overline{k}[\mu][Z, T]}{(1 - aZ^2 - Q(\mu)T^2, \mu)} = \frac{\overline{k}[Z, T]}{(1 - aZ^2 - a_0T^2)}$$
To see that B/μ is an integral domain, it suffices to show that $1 - aZ^2 - a_0T^2$ is irreducible. By taking $\mathfrak{p} = (z + \sqrt{a})$ as our prime ideal, its easy to see that $\mathfrak{p} | Z^2 - a$ and $\mathfrak{p}^2 \not| Z^2 - a$, hence by generalized Eisenstein, $1 - aZ^2 - a_0T^2$ is irreducible. This then implies that the only codimension 1 prime where μ vanishes is $\mathfrak{q} = (1 - aZ^2 - a_0T^2, \mu)$.

  In the local ring $B_{\mathfrak{q}}$, the relation $1 - aZ^2 - Q(\mu)T^2$ can be written as
$$1 - aZ^2 - a_0T^2 - \mu(\mu^3 + a_3\mu^2 + a_2\mu + a_1)T^2$$
Without loss of generality, we are assuming $a_1 \neq 0$. If it were zero, we would take the first non-zero coefficient and factor out the lowest power of μ attached to it. The existence of this constant term ensures that right-most term in the above equation

is $\mu(\text{unit})T^2$. Furthermore, T is a unit as well, hence $1 - aZ^2 - a_0T^2 \in (\mu)$ and μ is a uniformizer. Considering valuations, we can see that $1 - \sqrt{a}z \notin (1 - aZ^2 - a_0T^2)$ for if it were we would have $Z = \frac{1}{\sqrt{a}}$ in B/μ, which in turn would imply that B/μ is zero dimensional. It now remains to compute $v_{\mathfrak{q}}(\mu^2T)$.
Using the fact that μ is a uniformizer, we get

$$v_{\mathfrak{q}}(\mu^2T) = 2v_{\mathfrak{q}}(\mu) = 2$$

The vanishing of μ gives the smooth fiber F . We can then say that

$$\mathrm{Div}_{\overline{X}_{y\neq 0}}\left(\frac{y-\sqrt{a}z}{\mu^2T}\right) = -2F$$

By the argument we made for $\mu \neq 0$, the patch where $T \neq 0$ gives no new information, thus

$$\mathrm{Div}_{\overline{X}_{z\neq 0}}\left(\frac{y-\sqrt{a}}{t}\right) = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0 - 2F$$

From this we can conclude that $2F = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0$ in $\mathrm{Pic}\,\overline{X}$.

□

Proposition 19. *Let X be a Châtelet surface X defined over k. Then $\overline{X}$ is a blow-up of a Hirzebruch surface $\mathbb{F}_n$ at 4 points.*

Proof. First, observe that all vertical fibers (preimages of $\overline{\pi}$ over closed points of $\mathbb{P}^1_{\overline{k}}$) are linearly equivalent. Let $F = \overline{\pi}^{-1}(P)$ and $F' = \overline{\pi}^{-1}(P')$, then $F^2 = F.F' = 0$. From equation (4.1) we know that $E_i + \widetilde{E}_i - F$ is a principal divisor, hence in the Picard group, F and $E_i + \widetilde{E}_i$ are in the same class. We can now deduce that

$$0 = F^2 = F.(E_i + \widetilde{E}_i) = F.E_i + F.\widetilde{E}_i$$

Moreover, since the Galois action preserves the intersection pairing, we can conclude that $F.E_i = F.\widetilde{E}_i = 0$. Now, since intersecting with a principal divisor is 0 we have

$$0 = E_i.(E_i + \widetilde{E}_i - F) = E_i.E_i + E_i.\widetilde{E}_i + E_i.F = E_i^2 + 1$$

so $E_i^2 = -1$, for $i = 1, 2, 3, 4$. Intersecting with $\widetilde{E}_i$ we also see that $\widetilde{E}_i^2 = -1$. Take the four skew lines, E_i, and blow them down. Let $P_i \in E_i \cap \widetilde{E}_i$ in $\overline{X}$ and by abuse notation let P_i denote the image of P_i under f. By Proposition 14, we have that P_i is a smooth (multiplicity one) point on $\widetilde{E}_i$, thus $\widetilde{E}_i \cong \mathbb{P}^1_{\overline{k}}$. We are left with a $\mathbb{P}^1_{\overline{k}}$-bundle over $\mathbb{P}^1_{\overline{k}}$ which is none other than $\mathbb{F}_n$ for some n. □

Corollary 8. *Let $\overline{X}$ be a Châtelet surface defined over $\overline{k}$. Then*

$$\mathrm{Pic}\ X = \mathbb{Z}F \oplus \mathbb{Z}C_0 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4$$

with intersection theory given by

$$C_0.F = \widetilde{C}_0.F = 1 \quad (4.3)$$

$$C_0.E_i = \widetilde{C}_0.\widetilde{E}_i = 1 \quad (4.4)$$

$$C_0.\widetilde{E}_i = \widetilde{C}_0.E_i = 0 \quad (4.5)$$

$$E_i.\widetilde{E}_i = 1 \quad (4.6)$$

$$C_0.\widetilde{C}_0 = 0 \quad (4.7)$$

Proof. By realizing $\overline{X}$ as the blow-up of a Hirzebruch surface, we know that $C_0.F = 1$. Using equation (4.2) we can conclude that $F.\widetilde{C}_0 = 1$. Since $F \sim E_i + \widetilde{E}_i$ we can see that

$$1 = C_0.F = C_0.(E_i + \widetilde{E}_i) = C_0.E_i + C_0.\widetilde{E}_i$$

Looking at the equations that cut out C_0 and E_i we can see that $C_0.E_i = 1$. Equation (4.4) now follows as does (4.5) by a similar argument. Equation (4.6) is clear, and to deduce (4.7) observe that $\{y - \sqrt{a}z = 0, t = 0\}$ and $\{y + \sqrt{a}z = 0, t = 0\}$ cut out C_0 and $\widetilde{C}_0$ respectively. If the two curves were to intersect, their equations would have a common solution, but this solution must be $y = z = t = 0$. Since y, z, t are homogeneous coordinates, at least one of them must be non-zero. □

5. Main Results: The Brauer Group of a Châtelet Surface

In 1971, Iskovskikh provided an example of a Châtelet Surface that failed the Hasse principle [Isk71]. In this section, we compute the Brauer group of a Châtelet surface, and in doing so, will realize that classes of Châtelet surfaces that can fail the Hasse Principle are those in which the factorization of $P(\lambda)$ has a certain form [CTSSD87], greatly reducing a complex problem to a simpler one.

Theorem 7. *Let L denote the splitting field of P so that $L(\sqrt{a})$ is the splitting field of X. Assume $a \notin L^{\times 2}$. The Brauer group of X depends on the factorization of P with Brauer groups given by*

$$H^1(G_k, Pic(\overline{X})) = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & P(\lambda) \textit{has four rational roots} \\ \mathbb{Z}/2\mathbb{Z} & P(\lambda) \textit{has one irredicible quadratic factor} \\ \{0\} & \textit{otherwise} \end{cases}$$

The proof of Theorem 7 is explicit but requires several important reductions.

Lemma 4. *The surface $X_{k(\sqrt{a})}$ is rational. In particular, if $a \in k^{\times 2}$, then X is rational.*

Proof. Let $P(\lambda) = \prod g_j(\lambda)$ and let Q_{g_j} denote the closed point of degree $\deg(g_j)$ corresponding to g_j. The fiber over Q_{g_j} degenerates to the union of two lines over $k(\sqrt{a})$ which we label E_i and $\widetilde{E}_i$. Letting $\lambda = \alpha$ denote a smooth fiber F we have

$$\mathrm{Div}_{X_{k(\sqrt{a})}}\left(\frac{g_j(\lambda)}{(\lambda - \alpha)^{\deg(g_j)}}\right) = E_j + \widetilde{E}_j - (\deg(g_j)F)$$

Since $E_j.\widetilde{E}_j = \deg(g_j)$ and intersecting with a principal divisor is always zero we have

$$E_j.(E_j + \widetilde{E}_j - (\deg(g_j)F)) = E_j^2 - E_j.\widetilde{E}_j - (\deg(g_j))E_j.F = E_j^2 + \deg(g_j) = 0$$

thus E_j is a $(-\deg(g_j))$-curve. Moreover, each component of E_j is a (-1)-curve
For each factor of $P(\lambda)$ over $k(\sqrt{a})$ we have such a collection of curves. As in the proof of Proposition 21, by blowing down these skew groups of curves we obtain a Hirzebruch surface, which is rational by Corollary 7. □

One can see that if $a \in k^{\times 2}$, then X is rational hence has no Brauer-Manin obstruction.

Lemma 5. *The Brauer group of X modulo constant algebras is 2-torsion. That is*

$$H^1(G_k, \operatorname{Pic}\overline{X})[2] = H^1(G_k, \operatorname{Pic}\overline{X})$$

Proof. First, if $a \in k^{\times 2}$ then $\operatorname{Br} X = 0$, hence $H^1(G_k, \operatorname{Pic}\overline{X}) = 0$. If a is not a square in k then consider the subgroup $H = \operatorname{Gal}(\overline{k}/k(\sqrt{a}))$ of G_k, with $G_k/H \cong \operatorname{Gal}(k(\sqrt{a})/k) \cong \mathbb{Z}/2\mathbb{Z}$.

By Lemma 4

$$\frac{\operatorname{Br} X_{k(\sqrt{a})}}{\operatorname{Br} k(\sqrt{a})} \cong H^1(\operatorname{Gal}(\overline{k}/k(\sqrt{a})), \operatorname{Pic}\overline{X}) = 0$$

Restriction-corestriction implies that $Cor \circ Res = [2]$ and furthermore, this is the zero map on $H^1(G_k, \operatorname{Pic}\overline{X})$.

□

Lemma 6.

$$H^1(G_k, \operatorname{Pic}\overline{X}) \cong \frac{\left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k}}{\left(\operatorname{im}(\operatorname{Pic}\overline{X})^{G_k} \to \left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k}\right)}$$

Proof. Taking Galois cohomology with respect to the exact sequence

$$0 \to \operatorname{Pic}\overline{X} \xrightarrow{\times 2} \operatorname{Pic}\overline{X} \to \frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}} \to 0$$

we obtain the long exact sequence

$$0 \to (\operatorname{Pic}\overline{X})^{G_k} \xrightarrow{\times 2} (\operatorname{Pic}\overline{X})^{G_k} \xrightarrow{\psi} \left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k} \xrightarrow{\delta} H^1(G_k, \operatorname{Pic}\overline{X}) \xrightarrow{[2]} 0$$

Lemma 5 implies that $H^1(G_k, \operatorname{Pic}\overline{X})[2] = \mathrm{H}^1(G_k, \operatorname{Pic}\overline{X})$ by exactness so we obtain the desired isomorphism. □

Lemma 7. *A basis for $(\operatorname{Pic}\overline{X})^{G_k}$ is given by $\{F\}$*

Proof. Let L be the splitting field of $P(\lambda)$, so then $L(\sqrt{a})$ is the splitting field of X. Let $G = \operatorname{Gal}(L\sqrt{a})/k)$. It is easy to see that

$$(\operatorname{Pic} X_{L(\sqrt{a})})^G = \bigcap_{\sigma \in G} \ker(\sigma - \mathrm{id})$$

so in order to find a basis for $(\operatorname{Pic}\overline{X})^{G_k}$, we intersect bases for all the above eigenspaces.

We can easily see that $\sigma(F) = F$ for all $\sigma \in G$. To find a basis for $\bigcap_{\sigma \in G} \ker(\sigma - \mathrm{id})$, will show that $\ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \operatorname{Span}\{F\}$, where $\sigma_{\sqrt{a}}$ denotes the involution coming from $\operatorname{Gal}(k(\sqrt{a})/k)$ and so then $(\operatorname{Pic}\overline{X})^{G_k} = \operatorname{Span}\{F\}$. We fix the basis $\{F, C_0, E_1, E_2, E_3, E_4\}$ for $\operatorname{Pic}\overline{X}$ and represent the linear map $\sigma_{\sqrt{a}} - \mathrm{id}$ via a matrix in this basis.

$$\sigma_{\sqrt{a}} - id = \begin{bmatrix} 0 & -2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

From here it is immediate that this matrix has rank 5 with kernel spanned by F. This computation holds independent of the factorization of $P(\lambda)$ hence $(\operatorname{Pic}\overline{X})^{G_k} = \operatorname{Span}\{F\}$, and so $\operatorname{im}\Big((\operatorname{Pic}\overline{X})^{G_k} \to \big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}\Big) = \operatorname{Span}\{F\}$

□

Proposition 20. *A basis for* $\big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}$ *depends on the factorization of* $P(\lambda)$.

$$\Big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\Big)^{G_k} = \begin{cases} \operatorname{Span}\{F, E_1+E_2, E_2+E_3\} & P(\lambda) \textit{has four rational roots} \\ \operatorname{Span}\{F, E_1+E_2\} & P(\lambda) \textit{has one irredicible quadratic factor} \\ \operatorname{Span}\{F\} & \textit{otherwise} \end{cases}$$

Combining this with Lemma 7*, we obtain the main result of Theorem* 7*.*

Proof. Since $\operatorname{Pic}\overline{X} \cong \mathbb{Z}^6$ we obtain a basis for $\big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}$ by reducing the entries of the matrices $\sigma - \mathrm{id}$ modulo 2, finding bases for their respective kernels, and finally, computing a basis for their intersection. Each of the following cases can be easily computed by associating, to each $\sigma \in G$, its corresponding element of S_4. We note that $\sigma_{\sqrt{a}}(C_0) = \widetilde{C}_0 \neq C_0$ hence to determine the basis for each eigenspace, we need only look at how the E_i are permuted. Moreover, the permutations of these roots are given by $\operatorname{Gal}(L/k)$, hence it is enough to compute bases for $\ker(\sigma - \mathrm{id})$ where $\sigma \in \operatorname{Gal}(L/k)$. Given an automorphism τ, we denote its eigenspace of eigenvalue 1 by S_τ. A basis for S_τ is then given by the span of the following vectors

- F
- Sums of exceptional curves $\sum_{i\in I} E_i$ where $I = \{i|\tau(i) \neq i\}$
- Exceptional curves E_j where $\tau(j) = j$.

We note that for any $\tau \in \operatorname{Gal}(L/k(\sqrt{a}))$, $\tau - \mathrm{id}$ is represented as a matrix, and one can easily see that this matrix contains no entries divisible by 2, so the above method does in fact compute bases modulo $(2\operatorname{Pic}\overline{X})^{G_k}$.

We begin by noting that since the action of $\sigma_{\sqrt{a}}$ does not depend in the factorization of P, the basis for $\ker(\sigma_{\sqrt{a}} - \mathrm{id})$ modulo $(2\operatorname{Pic}\overline{X})^{G_k}$ will be the same for each case. Furthermore, we can observe that $\sigma_{\sqrt{a}}$ fixes any sum of the form $E_i + E_j$ because from equation (4.1)

$$\sigma_{\sqrt{a}}(E_i+E_j) = \widetilde{E}_i + \widetilde{E}_j = F - E_i + F - E_j = 2F + E_i + E_j = E_i + E_j \in \Big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\Big)^{G_k}$$

Reducing the matrix $(\sigma_{\sqrt{a}} - \mathrm{id})$ modulo 2 we obtain a linear map whose kernel is $\operatorname{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\}$. Furthermore, by equation (4.2) we know that $E_1 + E_2 + E_3 + E_4$ is linearly equivalent to $2F$ so $E_1 + E_2 + E_3 + E_4 = 0$ in $\big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}$. ($C_0$ is not fixed by any element of G). We can now conclude that $E_1 + E_2$ and $E_3 + E_4$ represent the same class in $\frac{\big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}}{\Big(\operatorname{im}(\operatorname{Pic}\overline{X})^{G_k} \to \big(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\big)^{G_k}\Big)}$

Case 1: $P(\lambda) = (\lambda^2 - b)(\lambda^2 - c)$

In this case, $\operatorname{Gal}(L/k) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and up to relabeling, is generated by (12) and (34) thus

$$S_{(12)} = \operatorname{Span}\{F, E_1+E_2, E_3, E_4\} \quad \text{and} \quad S_{(34)} = \operatorname{Span}\{F, E_1, E_2, E_3+E_4\}$$

hence

$$\left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k} = \operatorname{Span}\{F, E_1 + E_2, E_3 + E_4\}$$

We conclude that

$$\frac{\left(\frac{\operatorname{Pic}(\overline{X})}{2\operatorname{Pic}(\overline{X})}\right)^{G_k}}{\left(\operatorname{im}(\operatorname{Pic}(\overline{X})^{G_k} \to \left(\frac{\operatorname{Pic}(\overline{X})}{2\operatorname{Pic}(\overline{X})}\right)^{G_k}\right)} = \frac{\operatorname{Span}\{F, E_1 + E_2, E_3 + E_4\}}{\operatorname{Span}\{F\}} = \operatorname{Span}\{E_1 + E_2\} \cong \mathbb{Z}/2\mathbb{Z}$$

so

$$H^1(G_k, \operatorname{Pic}(\overline{X})) \cong \mathbb{Z}/2\mathbb{Z}$$

Case 2: $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 - b)$

In this case, $\operatorname{Gal}(L/k) \cong \mathbb{Z}/2\mathbb{Z}$ and is generated by (34), so $S_{(34)} = \operatorname{Span}\{F, E_1, E_2, E_3 + E_4\}$ and $\left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k} = \operatorname{Span}\{F, E_1 + E_2, E_3 + E_4\}$. By the same argument as the previous case, we conclude that

$$H^1(G_k, \operatorname{Pic}(\overline{X})) \cong \mathbb{Z}/2\mathbb{Z}$$

and is generated by the class of $E_1 + E_2$.

Case 3: $P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i)$

In this case, $\sigma_{\sqrt{a}}$ is the only non-trivial automorphism. This implies that

$$\left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k} = \ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \operatorname{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\}$$

It now follows that

$$H^1(G_k, \operatorname{Pic}(\overline{X})) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

and is generated by classes of $E_1 + E_2$ and $E_2 + E_3$.

Case 4: $P(\lambda) = (\lambda - \lambda_1)f(\lambda)$**, with** f **irreducible**

(a) $\underline{\operatorname{disc}(f) \in k^{\times 2}}$

Let $\tau = (123)$ be the generator of $\operatorname{Gal}(L/k) = A_3$. This means that $\ker(\tau - \mathrm{id}) = \operatorname{Span}\{F, E_1 + E_2 + E_3\}$ so intersecting with $\ker(\sigma_{\sqrt{a}} - \mathrm{id})$ we get a subspace of dimension 1, containing F hence

$$\left(\frac{\operatorname{Pic}\overline{X}}{2\operatorname{Pic}\overline{X}}\right)^{G_k} = \operatorname{Span}\{F\} \quad \text{and} \quad H^1(G_k, \operatorname{Pic}(\overline{X})) = \{0\}$$

(b) $\underline{\operatorname{disc}(f) \notin k^{\times 2}}$

Here, $\operatorname{Gal}(L/k) = \langle \rho = (12), \tau = (123)\rangle$. We get

$$\ker(\tau - \mathrm{id}) = \operatorname{Span}\{F, E_1 + E_2 + E_3\}, \quad \ker(\rho - \mathrm{id}) = \operatorname{Span}\{F, E_1, E_2 + E_3\}$$

so intersecting again with $\ker(\sigma_{\sqrt{a}} - id)$ we get a subspace of dimension 1 that must contain F, giving the same result.

Case 5: $P(\lambda)$ **irreducible**

Recall that the possibilities for $\mathrm{Gal}(L/k)$ are the transitive subgroups of S_4. These are the subgroups $S_4, A_4, D_4, \mathbb{Z}/4\mathbb{Z}$, and the unique normal subgroup that is isomorphic to $\left(\mathbb{Z}/2\mathbb{Z}\right)^2$.

Any transitive subgroup containing a 4-cycle will have eigenspace generated by F and $E_1+E_2+E_3+E_4$. No other sum of exceptional divisors (hence no individual exceptional divisor) is contained in this eigenspace, therefore no exceptional divisor is contained in $\cap_\sigma \ker(\sigma - id)$. From this we can conclude that

$$H^1(G_k, \mathrm{Pic}(\overline{X})) \cong \{0\}$$

whenever $\mathrm{Gal}(L/k) \cong S_4, D_4$, or $\mathbb{Z}/4\mathbb{Z}$.

If $\mathrm{Gal}(L/k) \geq \left(\mathbb{Z}/2\mathbb{Z}\right)^2 = \{(1), (12)(34), (13)(24), (14)(23)\}$, the non-trivial elements give eigenspaces $\mathrm{Span}\{F, E_1+E_2, E_3+E_4\}, \mathrm{Span}\{F, E_1+E_3, E_2+E_4\}$, and $\mathrm{Span}\{F, E_1+E_4, E_2+E_3\}$ respectively. Intersecting these three with $\ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \mathrm{span}\{F, E_1+E_2, E_2+E_3, E_3+E_4\}$ we see that only multiples of F and the element $E_1 + E_2 + E_3 + E_4$ lie in all 4. This shows that $H^1(G_k, \mathrm{Pic}(\overline{X}))$ is also trivial in this case.

Lastly, if $\mathrm{Gal}(L/k) \cong A_4$, then the non-trivial automorphisms correspond to 8 cycles of type $(1, 3)$ and 3 cycles of type $(2, 2)$. These correspond to eigenspaces of the form $\mathrm{Span}\{F, E_i + E_j + E_k\}$ for $i, j, k \in \{1, 2, 3, 4\}$. Assume that an element of the form $E_i + E_j + E_k \in \mathrm{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\} = \ker(\sigma_{\sqrt{a}} - \mathrm{id})$. Then in particular,

$$(E_i + E_j + E_k) + (E_i + E_j) = 2E_i + 2E_j + E_k = E_k \in \ker(\sigma_{\sqrt{a}} - \mathrm{id})$$

but $\sigma_{\sqrt{a}}(E_k) = \widetilde{E}_k \neq E_k$, hence $\bigcap_{\sigma\in G} \ker(\sigma - \mathrm{id}) = \mathrm{Span}\{F\}$ and we can conclude that $H^1(G_k, \mathrm{Pic}(\overline{X})) = \{0\}$ whenever P is irreducible, completing the proof. □

5.1. **An Example.** We now review a construction of Iskovskikh of a Châtelet surface that fails the Hasse priciple, following the exposition made in [Poo10, Section 8.2.5].
Let X be the Châtelet surface given by

$$y^2 + z^2 = (3 - \lambda^2)(\lambda^2 - 2)$$

over $\mathbb{Q}$. Given any regular, integral, Noetherian scheme X, we have an injection $\mathrm{Br}\, X \hookrightarrow \mathrm{Br}\, K(X)$ [Gro68a, Corollaire 1.10], where $K(X)$ denotes the function field of X. As explained in Remark 6 of section 2.2, given any two elements $a, b \in K(X)^\times$, one can define a quaternion algebra $(a, b) \in (\mathrm{Br}\, K(X))[2]$. Considering the quaternion algebra $\mathcal{A} = (3-\lambda^2, -1) \in \mathrm{Br}\, K(X)$, one can show that $\mathcal{A}$ lies in the subgroup $\mathrm{Br}\, X$ using residue homomorphisms. In particular, given an open cover $\{U_i\}$ of X, we have a sequence of injections $\mathrm{Br}\, X \hookrightarrow \mathrm{Br}\, U_i \hookrightarrow \mathrm{Br}\, K(X)$. Furthermore, given Brauer classes $\mathcal{B}_i \in \mathrm{Br}\, U_i$ whose images agree in $\mathrm{Br}\, K(X)$, one can conclude that they come from $\mathrm{Br}\, X$. This is a non-trivial fact, proven in [Poo10, Theorem 6.8.3].

Given any $g \in K(X)^\times$, the class of $(g, -1)$ is unaffected by multiplying g by a square or a norm from $k(\sqrt{-1})$. Let

$$\mathcal{B} = (\lambda^2 - 2, -1) \quad \text{and} \quad \mathcal{C} = (3/\lambda^2 - 1, -1)$$

From Proposition 5, we know that

$$[\mathcal{A}] + [\mathcal{B}] = [(y^2 + z^2, -1)] = 0 \quad \textit{and} \quad [\mathcal{A}] + [\mathcal{C}] = [((\frac{\lambda^2 - 3}{\lambda^2})^2, -1)] = 0$$

Since $2[\mathcal{A}] = 0$ we conclude that

$$[\mathcal{A}] = [\mathcal{B}] = [\mathcal{C}]$$

One can show [Poo10, Proposition 8.2.14] that there exists an open cover $\{U_\mathcal{A}, U_\mathcal{B}, U_\mathcal{B}\}$ of X on which $\mathcal{A}, \mathcal{B}, \mathcal{C}$ represent respective Brauer classes on each open set. It then follows that $[\mathcal{A}] \in \operatorname{Br} X$.

Each of these three representatives will be used to compute the set $X(\mathbb{A}_\mathbb{Q})^{\operatorname{Br}}$. To evaluate $\mathcal{A}$ at a point $P \in X(k)$ for any $k \supset \mathbb{Q}$, choose any of $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$ such that the rational function of λ is defined and non-zero at P, and replace the rational function by its value at P. For example, if P is defined and non-zero at $3 - \lambda^2$ then $\mathcal{A}(P) = (3 - \lambda(P)^2, -1) \in \operatorname{Br} k[2]$.

Since $\frac{\operatorname{Br} X}{\operatorname{Br} \mathbb{Q}} \cong \mathbb{Z}/2\mathbb{Z}$, we need only compute $X(\mathbb{A}_\mathbb{Q})^\mathcal{A}$ with $\mathcal{A}$, a generator of $\frac{\operatorname{Br} X}{\operatorname{Br} \mathbb{Q}}$. We will show that $\mathcal{A}$ gives a Brauer-Manin obstruction to the Hasse principle, that is, we show that $X(\mathbb{A}_\mathbb{Q})^\mathcal{A} \neq \emptyset$. We begin by noting that X has a $\mathbb{Q}_p$ point for every $p \leq \infty$, in other words

Proposition 21. $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$

Before we prove the proposition we recall the notion of a Hilbert symbol along with a basic property. For $v \in \Omega_k$ and $t, u \in k_v^\times$ we define the **Hilbert symbol** $(t, u)_v \in \{\pm 1\}$ by the rule $(t, u)_v = 1$ if and only if $x^2 - ty^2 = uz^2$ has a solution $(x, y, z) \neq (0, 0, 0) \in k_v^{\times 3}$. The main property of Hilbert symbols we will use is the following

Lemma 8. [Ser73, Chapter III] *Suppose that v is odd and that $v(t) = 0$. Then $(t, u)_v = -1$ if and only is $v(u)$ is odd and the image of t in $\mathbb{F}_v$ is a non-square.*

Proof. The fact that $X(\mathbb{R}) \neq \emptyset$ is obvious so we begin by assuming that v is 2-adic. By considering the univariate polynomial $f(\lambda) = (\lambda^2 - 2)(3 - \lambda^2)$ obtained by setting $y = z = 0$ we can see that

$$v(f(0)) > 2v(f'(0))$$

Hence the strong version of Hensel's Lemma [Lan94, Chapter II, Proposition 2.2] implies that there exists a root in $\mathbb{Q}_2$.

If v is odd and $v \neq v_3$, we aim to prove that $(-1, (x^2 - 2)(3 - x^2))_v = 1$ for $x \in \mathbb{Q}_p$. Note that this would imply the existence of a p-adic solution to $y^2 + z^2 = (\lambda^2 - 2)(3 - \lambda^2)$ for odd p not equal to 3. By Lemma 8, it is enough to find $x \in \mathbb{Q}_p$ such that $v_p((x^2 - 2)(3 - x^2))$ is even. Pick any $x \in \mathbb{Q}_p$ such that $v(x) \neq 0$, it follows that

$$v_p((x^2 - 2)(3 - x^2)) = v_p(x^2 - 2) + v_p(3 - x^2) = 2v_p(x^2) = 4v_p(x)$$

So X has a $\mathbb{Q}_p$ point for all $p \neq 3$.

When $p = 3$, the proof follows similarly. For $x \in \mathbb{Q}_3$ such that $v_3(x) < 0$ we have

$$v_3((x^2 - 2)(3 - x^2)) = 4v_3(x)$$

Hence X has a $\mathbb{Q}_3$ point. □

It now remains to compute $\mathrm{inv}_{v_p}(\mathcal{A}(P))$ and by implicit function theorem, it is enough to compute this for $P \in U(\mathbb{Q}_p)$ for any Zariski-dense open set $U \subset X$.

Lemma 9. *Let k be a local field and let X be smooth of dimension n over k. Let $U \subset X$ be a nonempty Zariski open set of X. Then $U(k)$ is analytically dense in $X(k)$.*

Proof. Take $P \in X(k)$. We show that we can find a sequence of points $P_i \in U(k)$ that converge to P. Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism of X, by [Poo10, Prop 3.5.48], there exists a Zariski open neighborhood V of P and an étale morphism $\varphi : V \to \mathbb{A}^n_k$ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathbb{A}^n_k \\ \downarrow{\scriptstyle \pi} & \swarrow & \\ \operatorname{Spec} k & & \end{array}$$

The étale map φ satisfies the hypothesis of the implicit function theorem, thus there exists *analytically* open neighborhoods $O_1 \ni P$ and O_2 of $V(k)$ and k^n respectively, such that φ induces a homeomorphism $\theta : O_1 \to O_2$.
Let $X \setminus U$ denote the Zariski complement of U and consider $(X \setminus U) \cap V$, which contains both P and O_1. Consider the closed set $C = \overline{\varphi(X \setminus U) \cap V)}$ of $\mathbb{A}^n_k$. Since C doesn't contain the image of U under φ, it is not all of $\mathbb{A}^n_k$ hence $\dim C < n$. Note that for any Zariski closed set $C = V(f_1, ..., f_n)$ we can construct a sequence of points P_i converging to $\varphi(P)$, by openness of O_2. Moreover, we can ensure that all $P_i \in O_2 \setminus C$. To see why, observe that the polynomials f_i are all continuous in the analytic toplogy. Since $V(f_i) = f_i^{-1}(0)$, we conclude that C is also closed in the analytic topology, hence $O_2 \setminus C$ is analytically open. It now follows that $\theta^{-1}(P_i) \in U(k)$ and by continuity of θ we have

$$\theta^{-1}(\lim P_i) = \lim \theta^{-1}(P_i) = P$$

$\square$

Proposition 22. *Fix a place p of $\mathbb{Q}$. Then for any $P \in X(\mathbb{Q}_p)$,*

$$\mathrm{inv}_{v_p}(\mathcal{A}(P)) = \begin{cases} 0 & p \neq 2 \\ \frac{1}{2} & p = 2 \end{cases}$$

Proof. Let X_0 be the affine surface in $\mathbb{A}^3$ given by $y^2 + z^2 = (3 - \lambda^2)(\lambda^2 - 2)$. Since X is smooth, Lemma 9 shows that $X_0(\mathbb{Q}_p)$ is p-adically dense in $X(\mathbb{Q}_p)$. By Lemma 3, $\mathrm{inv}_{v_p} \circ \mathrm{ev}_{\mathcal{A}}$ is a continuous function on $X(\mathbb{Q}_p)$, hence it suffices to prove the result for $P \in X_0(\mathbb{Q}_p)$.
For each $P \in X_0(\mathbb{Q}_p)$, $\mathcal{A}(P)$ defines a quaternion algebra in $\mathrm{Br}(\mathbb{Q}_p)$. In particular, Remark 6 gives a description of $\mathcal{A}$ as a cyclic algebra where $j^2 = -1$ and $i^2 = x$ with $x = 3 - \lambda^2, \lambda^2 - 2$, or $3/\lambda^2 - 1$. Proposition 7 allows us to easily compute the image of $\mathcal{A}(P)$ under the invariant map and tells us that $\mathrm{inv}_{v_p}(\mathcal{A}(P)) = 0$ if and only if $x(P) \in N_{\mathbb{Q}_p(i)/\mathbb{Q}_p}(\mathbb{Q}_p(i)^\times)$. In other words,

$$\mathrm{inv}_{v_p}(\mathcal{A}(P)) = \begin{cases} 0 & \mathcal{A}(P) \text{is split} \\ \frac{1}{2} & \mathcal{A}(P) \text{is non-split} \end{cases}$$

Case 1: $p \notin \{2, \infty\}$

If $v_p(\lambda) < 0$ then $v_p(3/\lambda^2 - 1) = \min\{v_p(3) - 2v_p(\lambda), 0\}$. Note that $v_p(3) = 0$ or 1 but $-2v_p(\lambda) > 0$ hence $v_p(3) - 2v_p(\lambda) > 0$ and $v_p(3/\lambda^2 - 1) = 0$. This implies that $3/\lambda^2 - 1 \in \mathbb{Z}_p^\times$.
If $v_p(\lambda) \geq 0$ then

$$0 = v_p(1) = v_p(3 - \lambda^2 + \lambda^2 - 2) \geq \min\{v_p(3 - \lambda^2), v_p(\lambda^2 - 2)\}$$

which implies that

$$\min\{v_p(3), v_p(\lambda^2)\} \leq v_p(3 - \lambda^2) \leq 0 \quad \text{or} \quad \min\{v_p(2), v_p(\lambda^2)\} \leq v_p(\lambda^2 - 2) \leq 0$$

hence one of $3 - \lambda^2$ or $\lambda^2 - 2$ are in $\mathbb{Z}_p^\times$.
Now, we can recognize $\mathcal{A}(P)$ as an element of $\operatorname{Br}(\mathbb{Z}_p)$ via the definition of an Azumaya algebra over the ring $\mathbb{Z}_p$. It is given by a $\mathbb{Z}_p$-algebra A_p that is free and of finite rank as a $\mathbb{Z}_p$-module, such that $A_p \otimes_{\mathbb{Z}_p} k(x)$ is a central simple algebra over $k(x)$, where $k(x)$ denotes the residue field of $x \in \mathbb{Z}_p$. We can now see that $\operatorname{ev}_{\mathcal{A}}(P) = \mathcal{A}(P) = (u, -1)$ with $u \in \mathbb{Z}_p^\times$. By considering the Azumaya algebra $A_p \coloneqq \mathbb{Z}_p \otimes \mathbb{Z}_p i \otimes \mathbb{Z}_p j \otimes \mathbb{Z}_p ij$, with multiplication defined as for $\mathcal{A}(P)$, we have that $A_p \in \operatorname{Br} \mathbb{Z}_p$. Moreover, $A_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{A}(P)$ thus A_p maps to $\mathcal{A}(P)$ under $\operatorname{Br} \mathbb{Z}_p \to \operatorname{Br} \mathbb{Q}_p$ and $\operatorname{Br} \mathbb{Z}_p = 0$ by Corollary 4. This implies that $\mathcal{A}(P)$ is trivial in $\operatorname{Br} \mathbb{Q}_p$ hence $\operatorname{inv}_{v_p}(\mathcal{A}(P)) = 0$.

Case 2: $p = \infty$

Any $P \in X_0(\mathbb{R})$ satisfies

$$(3 - \lambda(P)^2)(\lambda(P)^2 - 2) = y^2 + z^2 > 0$$

hence either

$$(3 - \lambda(P)^2) \geq 0 \quad \text{and} \quad (\lambda(P)^2 - 2) \geq 0$$

or

$$(3 - \lambda(P)^2) \leq 0 \quad \text{and} \quad (\lambda(P)^2 - 2) \leq 0$$

The latter isn't possible for if it were we could have $3 \leq \lambda(P)^2 \leq 2$. Since the former holds then

$$(3 - \lambda(P)^2), (\lambda(P)^2 - 2) \in \mathbb{R}_{>0} = \mathrm{N}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$$

implying that $\mathcal{A}(P) = 0 \in \operatorname{Br} \mathbb{R}$ by Proposition 1, thus $\operatorname{inv}_{v_\infty}(\operatorname{ev}_{\mathcal{A}}(P)) = 0$.

Case 3: $p = 2$

Let $P \in X(\mathbb{Q}_2)$ and recall that an element $x \in \mathbb{Z}_2$ is not of the form $a^2 + b^2$ if $x \equiv -1 \pmod 4$. We consider three possibilities.
If $v_2(\lambda(P)) > 0$ then $\lambda(P)^2 \equiv 0 \pmod 4$ hence $3 - \lambda(P)^2 \equiv 3 \equiv -1 \pmod 4$
If $v_2(\lambda(P)) = 0$ then $\lambda(P)^2 \equiv 1 \pmod 4$ hence $\lambda(P)^2 - 2 \equiv -1 \pmod 4$
If $v_2(\lambda(P)) < 0$ then $\frac{1}{\lambda(P)^2} \equiv 0 \pmod 4$ hence $3/\lambda(P)^2 - 1 \equiv -1 \pmod 4$

We can see that in all three cases, $\mathcal{A}(P) = (x, -1)$ where x is not a norm from $\mathbb{Q}_2(i)/\mathbb{Q}_2$, hence by Proposition 1, $\mathcal{A}(P)$ is non-split and $\operatorname{inv}_{v_2}(\mathcal{A}(P)) = \frac{1}{2}$.

□

Corollary 9.

$$X(\mathbb{A}_k)^{\mathcal{A}} = \emptyset$$

Proof. From Proposition 22, it follows that $\sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = \frac{1}{2}$ so

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(P_v) \in X(\mathbb{A}_k) \colon \sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 0\} = \emptyset$$

This impies that $X(\mathbb{Q}) = \emptyset$. □

6. Potential Hasse Principle Violations for Châtelet Surfaces

6.1. **Remaining Questions.** Having seen an example of a Châtelet surface that fails the Hasse principle, it is natural to wonder what can be said about Châtelet surfaces X/k such that $X(\mathbb{A}_k) = \emptyset$. In particular, given such a surface, can we find an extension L/k such that $X(L) = \emptyset$ while $X(\mathbb{A}_L)^{\mathrm{Br}} \neq \emptyset$? Since all failures of the Hasse principle are explained by the Brauer-Manin obstruction [CTSSD87], this is precisely what one would need to conclude that X fails the Hasse principle over L.

Definition 13. Given a nice variety $X/\mathbb{Q}$ such that $X(\mathbb{Q}) = \emptyset$ we say X is a **potential Hasse principle violation** if there exists an extension $L/\mathbb{Q}$ such that $X(L) = \emptyset$ and $X(L_w) \neq \emptyset$ for all valuations w extending the discrete valuations v_p on $\mathbb{Q}$.

In 2011, Pete Clark examined questions involving potential Hasse principle violations for curves over global fields [Cla09]. In particular, he conjectured the following

Conjecture 1. Every curve C/k of genus ≥ 2 with $X(k) = \emptyset$ is a potential Hasse principle violation

This conjecture is still open. More recently, related questions have been asked for surfaces. In particular

Question 1. Is every Châtelet surface X/k with $X(\mathbb{A}_k) = \emptyset$ a potential Hasse principle violation?

In 2019, Bianca Viray and Brendan Creutz investigated a related question and obtained a positive result.

Theorem 8. *(Creutz-Viray) Let X/k be a Châtelet surface. There exists a finite set of places $S \subseteq \Omega_k$ and a set of local quadratic extensions L_v/k_v for all $v \in S$, such that if F/k is a quadratic extension with $F_v = L_v$ for all $v \in S$ then $X(\mathbb{A}_F)^{\mathrm{Br}} = X(\mathbb{A}_F) \neq \emptyset$.*

This theorem lends itself to the following question

Question 2. Can one characterize the Châtelet surfaces that are potential Hasse priciple violations?

Additionally, (Creutz-Viray) posed the question for quadratic extensions. That is, for what quadratic extensions can one obtain a global point? It was proven that this can be done for almost all of them, but the ones in which you can not, remain to be determined. This result marks the obvious starting point in answering question 2.

Remark 10. Theorem 6 makes a considerable reduction in approaching this problem. In particular, the results tell us that the Châtelet surfaces that we aim to characterize are given by

$$y^2 - az^2 = f(\lambda)g(\lambda)$$

where f, g are irreducible quadratics.

References

[Bea78] Arnaud Beauville, *Complex Algebraic Surfaces*, Press Syndicate of the University of Cambridge, Cambridge, 1978. ↑6, 14, 4.1

[Châ59] F. Châtelet, *Points rationnels sur certaines courbes et surfaces cubiques*, L'Enseignement Mathématique. Revue Internationale. IIe Serie (1959). ↑4.2

[Cla09] L. Pete Clark, *On the Hasse principle for Shimura curves*, Israel J. Math. 171 (2009). ↑6.1

[CTSSD87] J.L. Colliot-Théléne, J.J. Sansuc, and H.P.F. Swinnnerton-Dyer, *Intersection of two quadrics and Cht̂elet surfaces*, J. Reine Angew. Math **374** (1987). ↑1, 5, 6.1

[Gro68a] Alexander Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, 1968. ↑5.1

[Gro68b] ———, *Le groupe de Brauer. III. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, 1968. ↑4

[GS17] Philippe Gille and Tamás Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambidge University Press, Cambridge, 2017. ↑2.1, 2, 3, 2, 5, 6

[Gui18] Pierre Guillot, *A Gentle Course in Local Class Field Theory*, Cambridge University Press, Cambridge, 2018. ↑6

[Har70] Robin Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Springer-Verlag, New York-Heidelberg, 1970. ↑4.2

[Har77] ———, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977. ↑4.1, 11, 12, 13, 5, 15, 16

[Isk71] V.A. Iskovskih, *A counterexample to teh Hasse principle for systems of two quadratic forms in five variables*, Mat. Zametki **10** (1971). ↑1, 5

[Lan94] Serge Lang, *Algebraic Number Theory*, Springer-Verlag, New York, 1994. ↑5.1

[Man71] Yu. I Manin, *Le groupe de Brauer-Grothendieck en g eom etrie diophantienne*, Actes du Congr'es International des Math ematiciens (Nice, 1970 (1971). ↑1

[Man74] Yuri Manin, *Cubic forms: algebra, geometry, arithmetic*, North- Holland Publishing Co., Amsterdam, 1974. ↑3.4

[Mil80] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980). ↑8, 2

[Mil13] ———, *Class Field Theory (v4.02)*, 2013. available at www.jmilne.org/math/. ↑2.3

[Neu91] Jurgen Neukirch, *Algebraic Number Theory*, Springer-Verlag, Heidelberg, 1991. ↑

[NSW08] J Neukirch, A Schmidt, and K Wingberg, *Cohomology of number fields*, Springer-Verlag, Berlin, 2008. ↑3.3

[Poo00] Bjorn Poonen, *Computing rational points on curves*, I Number Theory for the millenium, III (2000). ↑

[Poo09] ———, *Existence of rational points on smooth projective varieties*, J. Eur. Math. Soc. (JEMS) **11** (2009). ↑

[Poo10] ———, *Rational Points on Varieties*, American Mathematical Society, Rhode Island, 2010. ↑3.1, 3.2, 3.3, 5.1, 5.1

[Ser73] Jean-Pierre Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973. ↑8

[Ser91] ______, *Local Fields*, Springer-verlag, New York, 1991. ↑2.2

[Vir10] Bianca Viray, *The algebraic Brauer-Manin obstruction on Chatelet surfaces, degree 4 del Pezzo surfaces, and Enriques surfaces* (2010). ↑3

[Voi19] John Voight, *Quaternion Algebras*, 2019. available at https://math.dartmouth.edu/ jvoight/quat-book.pdf. ↑2.1, 2.2